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THE TRIGONOMETRY OF ONE ANGLE.

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THE TRIGONOMETRY OF ONE ANGLE

UNIV. OF TORONTO

BY THE

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PREFACE.

THE elementary ideas of the Trigonometrical Ratios are very simple, and familiarity with them is of very great advantage in the study of Elementary Mechanics both Theoretical and Practical.

There must therefore be many students, who have little leisure for the study of pure Mathematics, who would find it worth their while to acquire a knowledge of the Trigonometry of one angle.

Moreover the University of Cambridge requires in the Mechanics section of the Previous Examination and in the General Examination a knowledge of the Trigonometry of one angle.

I have therefore condensed and arranged the following pages from my Elementary Trigonometry in the hope of providing a text book suitable for the large class of students above referred to.

A few of the articles have been re-written. The

application of the signs + and − seems to follow naturally from the consideration of Angles and Lines as *displacements* of direction and of distance respectively.

A set of Examination Papers for Exercise are appended. Solutions are given of alternate Papers.

Any suggestions for the improvement of the book will be gratefully received.

J. B. L.

CAIUS COLLEGE,
February, 1891.

NOTE.

Chapters IX. and X. are not required in the Cambridge Previous Examination.

Chapters VII., VIII., IX., X. and probably VI. are not required in the Cambridge General Examination.

Chapters VII. and X. are not required by Candidates for the Advanced Stage of Theoretical Mechanics in the South Kensington Science and Art Examination.

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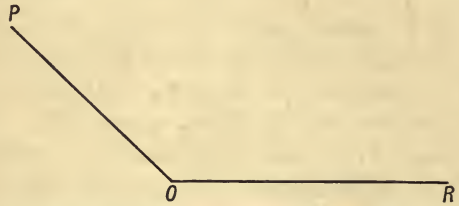
THE TRIGONOMETRY OF ONE ANGLE.

CHAPTER I.

1. DEFINITION. The **angle** between two lines *OR, OP* is the **amount of turning** about the point *O* which one of the lines *OP* has gone through in turning from the direction *OR* into the direction *OP*.



Thus in Trigonometry the angle *ROP* indicates, not necessarily the present *inclination* of the two lines *OR, OP*, nor the *least* amount of turning required to bring the line *OR* into the direction *OP*, but a certain amount of turning which *OP* has gone through since a certain instant when it had its *initial* direction *OR*.

2. An angle is a *displacement of direction.*

Example i. A *complete revolution* of the line *OP* from the direction *OR* and the direction *OR* again is a trigonometrical angle. Such a complete revolution may be called an angle of four right angles.

Example ii. *What is the* angle *described by the minute hand of a clock between* 3 o'clock *and* a quarter to 5?

The hand makes a complete revolution + three quarters of a complete revolution.

The angle is $1\frac{3}{4}$ of 4 right angles; that is, 7 right angles.

3. An angle may be described about a point in either of *two directions.*

You may cause a line to turn about a point in the direction in which the hands of a clock turn, or you may cause it to turn in the opposite direction.

It is convenient to refer to these two directions as **clockwise** and **counter-clockwise** respectively.

Fig. I.

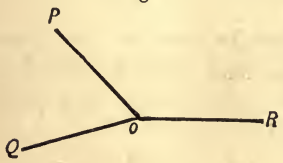
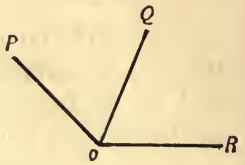


Fig. II.



4. By the **addition** of two angles $ROP + POQ$, we indicate an angle traced out by the revolving line which turns first through one of the angles ROP, and then on in the *same direction* through the other angle POQ. [Fig. I.]

For *example* $+1$ right angle $+\frac{2}{3}$ right angle $= (+1+\frac{2}{3})$ right angle.

By the **subtraction** of one angle from another $(+ROP - POQ)$ we indicate an angle traced out by the revolving line, turning through one angle ROP and then back in the *opposite direction* through the other angle POQ.

For *example*

$+1$ right angle $-\frac{2}{3}$ right angle $= (+1-\frac{2}{3})$ right angle $= \frac{1}{3}$ right angle.

Moreover, if we interpret the signs + and − as applied to angles to indicate opposite directions of revolution we may subtract a greater angle from the less.

For *example*

-1 right angle $+\frac{2}{3}$ right angle $= (-1+\frac{2}{3})$ right angle $= -\frac{1}{3}$ right angle.

Where $-\frac{1}{3}$ right angle indicates that the revolving line has been turned through $\frac{1}{3}$ right angle in the opposite direction to that in which it turned to trace out $+\frac{2}{3}$ right angle.

5. In what follows we shall usually take + as indicating the *counter-clockwise* direction and − as indicating the *clockwise* direction.

6. The angle ROP may be the *geometrical* representative of an unlimited number of Trigonometrical angles.

(i) The angle ROP may represent the angle less than two right angles as in Euclid.

In this case OP has turned from the position OR into the position OP by turning about O in the direction *contrary* to that of the hands of a watch.

(ii) The angle ROP may represent the angle described by OP in turning from the position OR into the position OP in the *same* direction as the hands of a watch.

In the first case it is usual to say that the angle ROP is described in the *positive* direction, in the second that the angle is described in the *negative* direction.

(iii) The angle ROP may be the geometrical representation of any of the Trigonometrical angles formed by any number of complete revolutions in the *positive* or in the *negative* direction, added to either of the first two angles. (We shall return to this subject later on.)

NOTE. The student will find that when an angle is referred to, the symbol π is often used to denote **two right angles**.

EXAMPLES. I.

Give a geometrical representation of each of the following angles, the starting line being drawn in each case from the turning point towards the right.

1. $+3$ right angles.
2. $+5$ right angles.
3. $+4\frac{1}{2}$ right angles.
4. $+7\frac{1}{4}$ right angles.
5. -1 right angle.
6. $10\frac{2}{3}$ right angles.
7. $-10\frac{1}{3}$ right angles.
8. $+4$ right angles.
9. -4 right angles.
10. $4n$ right angles.
11. $(4n+2)$ right angles.
12. $-(4n+\frac{1}{2})$ right angles.
13. The angle 3π.
14. The angle $\frac{2}{3}\pi$.

Rectangular Measure.

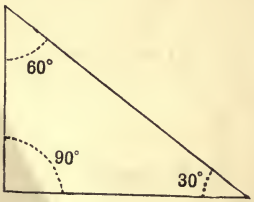
7. Angles are always measured *in practice* with the **right angle** (or part of the right angle) as unit.

8. The reasons why the right angle is chosen for a unit are :

(i) All right angles are equal to one another.
(ii) A right angle is practically easy to draw.
(iii) It is an angle whose size is very familiar.

9. The right angle is a large angle, and it is therefore subdivided for practical purposes.

10. For the purposes of practical measurement the right angle is divided into 90 equal parts, each of which is called a **degree**; each degree is subdivided into 60 equal parts, each of which is called a **minute**; and each minute is again subdivided into 60 equal parts, each of which is called a **second.**



Instruments used for measuring angles are subdivided accordingly; and the size of an angle is known when, with such an instrument, it has been observed that the angle contains a certain number of degrees, and a certain number of minutes beyond the number of complete degrees, and a certain number of seconds beyond the number of complete minutes.

Thus an angle might be recorded as containing 79 degrees + 18 minutes + 36·4 seconds.

Degrees, minutes, and seconds are indicated respectively by the symbols °, ′, ″, and the above angle would be written

$$79° \,.\, 18' \,.\, 36{\cdot}4''.$$

The expression 'the angle A,' when A stands for some number, is an abbreviation for 'the angle $A°$.'

When it is clearly understood that the unit is a degree there is no difficulty caused by omitting the symbol °.

NOTE. It was pointed out by the French at the end of the eighteenth century that it would be convenient to subdivide the right angle into decimal parts. It was proposed to call the hundredth part of a right angle a **grade**, the hundredth part of a grade, a **minute**, and the hundredth part of a minute a **second**. But as all calculations and results had hitherto been recorded in degrees etc., the disadvantage of having to alter all books and records of all kinds has proved an effectual hindrance to the adoption even by the French of the decimal (or as it is sometimes called the *French*) subdivision of the right angle.

Example. Express 3°. 45′. 27″ as the decimal of a right angle,

$$\begin{array}{r} 60)27'' \text{ seconds} \\ 60)45{\cdot}45 \text{ minutes} \\ 90)3{\cdot}7575 \text{ degrees} \\ {\cdot}04175 \text{ right angles} \end{array}$$

and ·04175 = 4 grades 17 (French) minutes 50 (French) seconds, and is written thus $4^g\ 17'\ 50''$.

EXAMPLES. II.

Express each of the following angles (i) as the decimal of a right angle, (ii) in grades, minutes, and seconds;

1.	8° 15′ 27″.	**2.**	6° 4′ 30″.	**3.**	97° 5′ 15″.
4.	16° 14′ 19″.	**5.**	132° 6′.	**6.**	49°.

Express in degrees, minutes and seconds,

7.	$1^g\ 37'\ 50''$.	**8.**	$8^g\ 75'$.	**9.**	$170^g\ 45'\ 35''$.
10.	$24^g\ 0'\ 25''$.	**11.**	$18^g\ 1'\ 15''$.	**12.**	35^g.

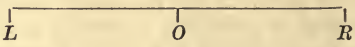
Draw a figure giving the position of the revolving line after it has turned through each of the following angles;

13.	270°.	**14.**	370°.	**15.**	425°.	**16.**	200°.
17.	590°.	**18.**	−45°.	**19.**	−330°.	**20.**	−390°.
21.	−480°.	**22.**	−630°.	**23.**	−600°.	**24.**	−750°.

11. DEF. By a linear **displacement** OR we indicate the moving of a point P along the line OR, *from* O *to* R.

Thus a *displacement* has *direction* as well as length.

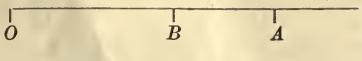
12. If we fix our attention on a particular line LOR, we have the choice of *two* directions in which a displacement from a starting point O along that line may be made.



A displacement may be made in the direction from O towards R or in the direction from O towards L.

13. By the *addition* of two displacements such as $(+a$ feet $+b$ feet), we indicate the moving of a point along a certain line from a certain point O through the distance $OA = a$ feet and then on, in the *same direction*, through the distance $AB = b$ feet.

By the *subtraction* of two displacements such as $(+a$ feet $-b$ feet), we indicate the moving of a point along a certain line from a certain point O, through the distance $OA = a$ feet and then back in the *opposite direction* through the distance $AB = -b$ feet.



Moreover if we interpret the signs + and − as applied to displacements to indicate opposite directions of motion we may subtract a greater displacement from a less.

Example. Let LR be a straight line parallel to the printed lines in the page, and let the positive direction of displacement be from L towards R.

Let A, B, C, D, E be points in LR, such that the measures of AB, BC, CD, DE, are 1, 2, 3, 4 respectively.

Find the algebraical representation of

(i) $AC+CB$ (ii) $AD+DC-BC$.

(i) The algebraical representation of AC is $+3$,
the algebraical representation of CB is -2.
Hence that of $AC+CB$ is $+3-2$; that is, $+1$.

(ii) The algebraical representation of AD is $+6$, that of DC is -3, and that of BC is $+2$.

Therefore that of $AD+DC-BC=6-3-2=+1$.

This is equivalent to that of AB.

EXAMPLES. III.

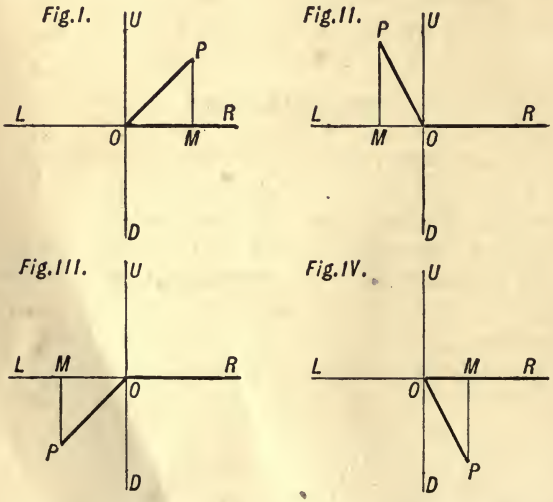
In the above figure, find the algebraical representation of

1. $AB+BC+CD$.
2. $AB+BC+CA$.
3. $BC+CD+DE+EC$.
4. $AD-CD$.
5. $AD+DB+BE$.
6. $BC-AC+AD-BD$.
7. $CD+DB+BE$.
8. $CD-BD+BA+AC+CE$.

CHAPTER II.

The Trigonometrical Ratios.

14. Let any angle be described by a line OP called the **revolving line**, turning about a chosen point O called the **origin** and starting from a chosen position OR called the **initial line**.



Take any point P in the revolving line and draw from P a line PM perpendicular to the initial line (produced, when necessary).

In this way we get a right-angled triangle OPM.

We have the direction OR fixed and we have a line perpendicular to OR also fixed in direction.

We agree that the + direction along OR shall be from O to R and that the + direction perpendicular to OR shall be from O to U where the angle ROU is a right angle drawn in the positive direction of rotation.

Then in fig. I., the displacement OM is positive.
in fig. II., ,, OM is negative.
in fig. III., ,, OM is negative.
in fig. IV., ,, OM is positive.

Again in fig. I., the displacement MP is positive.
in fig. II., ,, MP is positive.
in fig. III., ,, MP is negative.
in fig. IV., ,, MP is negative.

The direction of the displacement OP is indicated by the magnitude of the angle and the positive direction along the line OP is from O to P.

15. In the right-angled triangle obtained from an angle as in Article 14, the side MP *opposite* the origin or angular point is called the **perpendicular;**

the side OP opposite the right angle is called the **hypotenuse;**

the third side OM is called the **base.**

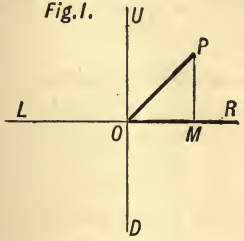
But it must be understood that the *perpendicular* is the *displacement* MP; that is, it is the displacing of a point from M to P and it will have the sign + or the sign − belonging to it according as the displacement is in the direction from O to U or in the direction from U to O.

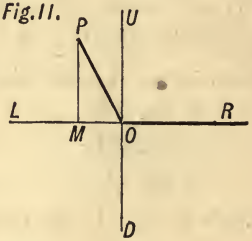
Also the *base* is the *displacement* OM.

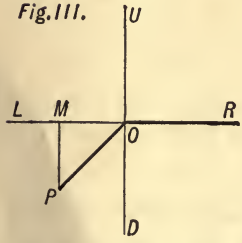
The *hypotenuse* is the *displacement* OP.

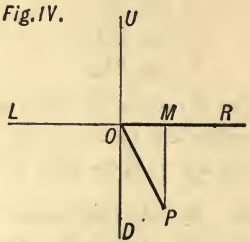
16. *To define the* **sine, cosine** *and* **tangent** *of an angle.*

Let OR be the initial line and let the angle $A°$ be described about the point O by the revolving line OP starting from OR.









Let OU be the direction of OP after it has described a right angle in the *positive* direction of revolution; then the positive directions of displacement parallel to OR, OU and OP are respectively from O to R, from O to U and from O to P.

On the revolving line take any point P; from P draw PM perpendicular to RO (produced, if necessary); then

(i) the **sine** of A is the ratio $\frac{MP}{OP}\left[=\frac{\text{perpendicular}}{\text{hypotenuse}}\right]$,

(ii) the **cosine** of A is the ratio $\frac{OM}{OP}\left[=\frac{\text{base}}{\text{hypotenuse}}\right]$,

(iii) the **tangent** of A is the ratio $\frac{MP}{OM}\left[=\frac{\text{perpendicular}}{\text{base}}\right]$.

MP, OM, and OP being *displacements* and carrying the sign indicated by the direction of the letters in the figure.

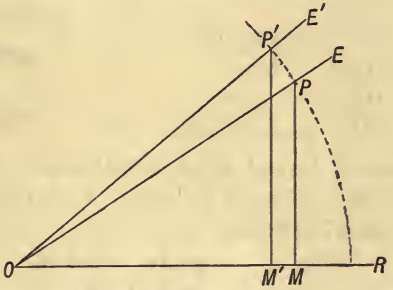
NOTE. The student should observe carefully

(i) that each *ratio*, such as $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is a mere *number;*

(ii) that, as we shall prove in Art. 24, these ratios remain unchanged as long as the angle remains unchanged;

(iii) that if the angle be altered ever so slightly, there is a consequent alteration in the value of these ratios;

[For, let ROE, ROE' be two angles which are nearly equal;



Let $OP = OP'$; then OM is *not* $= OM'$, and therefore the ratios $\frac{OM}{OP}$ and $\frac{OM'}{OP'}$ are *not* equal: also MP is *not* $= M'P'$ and therefore the ratios $\frac{MP}{OP}$ and $\frac{M'P'}{OP'}$ are *not* equal.]

(iv) that by giving **names** to these ratios we are enabled to apply the methods of Algebra to the Geometry of Euclid VI.

The student is recommended to pay careful attention to the definitions of Art. 16. He should be able to write them out in the exact words in which they are printed.

17. If A stand for the angle ROE, these ratios are called sine A, cosine A and tangent A, and are usually abbreviated thus:

$$\sin A, \qquad \cos A, \qquad \tan A.$$

18. There are three other Trigonometrical Ratios, formed by *inverting* the sine, cosine and tangent respectively, which are called the cosecant, secant, and cotangent respectively.

19. *To define the three other Trigonometrical Ratios of any angle.*

The same construction and figure as in Art. 16, being made, then the ratio

(iv) $\frac{OP}{MP} = \frac{\text{hypotenuse}}{\text{perpendicular}}$ is called the **cosecant** of the angle *ROE*.

(v) $\frac{OP}{OM} = \frac{\text{hypotenuse}}{\text{base}}$ „ **secant** „

(vi) $\frac{OM}{MP} = \frac{\text{base}}{\text{perpendicular}}$ „ **cotangent** „

20. Thus if A stand as before for the angle ROE, these ratios are called cosecant A, secant A, and cotangent A. They are abbreviated thus,

$$\text{cosec } A, \qquad \sec A, \qquad \cot A.$$

21. From the definition it is clear that

$$\text{cosec } A = \frac{1}{\sin A},$$

$$\sec A = \frac{1}{\cos A},$$

$$\cot A = \frac{1}{\tan A}.$$

22. The above definitions **apply to an angle of any magnitude.** (We shall return to this subject later on.)

For the present the student may confine his attention chiefly to angles which are each less than a right angle.

23. The powers of the Trigonometrical Ratios are expressed as follows:

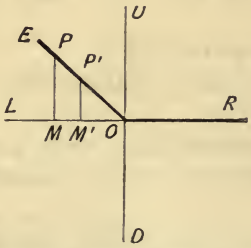
$(\sin A)^2$, i.e. $\left(\frac{\text{perpendicular}}{\text{hypotenuse}}\right)^2$, is written $\sin^2 A$,

$(\cos A)^3$, i.e. $\left(\frac{\text{base}}{\text{hypotenuse}}\right)^3$, is written $\cos^3 A$,

and so on.

The student must notice that '$\sin A$' is a *single symbol*. It is the *name* of a *number*, or *fraction*, belonging to the angle A; and if it be at any time convenient, we may denote *sin A* by a *single letter*, such as s or x. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, that is for $(\sin A) \times (\sin A)$. Such abbreviations are used because they are *convenient*.

24. *The Trigonometrical Ratios are always the same for the same angle.*



Take any angle ROE; let P, P' be any two points in OE one of the lines containing the angle. Draw PM, $P'M'$ perpendiculars to OR.

Then the triangles OMP, $OM'P'$, each contain a right angle, and they have the angle at O common; therefore their third angles must be equal.

Thus the three triangles are equiangular.

Therefore the ratios $\frac{MP}{OP}$, $\frac{M'P'}{OP'}$, are equal. (Eu. VI. 4.)

But each of these ratios is $\frac{\text{perpendicular}}{\text{hypotenuse}}$ *with reference to the angle at O*; that is, they are each *sin ROE*.

Therefore sin ROE is always the same.

25. A similar proof holds good for each of the other ratios.

Also if two angles are equal, it is clear that the numerical values of their Trigonometrial Ratios will be the same.

We have already shown (Note, p. 11), that the values of these ratios are different for different angles.

Hence for each particular value of A, $\sin A$, $\cos A$, $\tan A$, etc. have *definite numerical values.*

Example. We shall prove (Art. 32) that

$$\sin 30° = \tfrac{1}{2} = \cdot 5, \quad \cos 30° = \frac{\sqrt{3}}{2} = \cdot 8660\ldots, \quad \tan 30° = \frac{1}{\sqrt{3}} = \cdot 577\ldots$$

26. In the following examples, in which the angles considered are each less than a right angle, the student should notice

(i) the *angle* referred to,

(ii) that there is a *right angle* in the same triangle as the angle referred to,

(iii) the *perpendicular*, which is opposite the angle referred to, and is perpendicular to one of the lines containing the angle,

(iv) the *hypotenuse*, which is opposite the right angle,

(v) the *base*, the third side of the triangle.

Example. In the second figure on the next page, in which BDA is a right angle, find $\sin DBA$ and $\cos DBA$.

In this case

(i) DBA is the *angle*.

(ii) BDA is a *right angle* in the same triangle as the angle DBA.

(iii) DA is the *perpendicular*, for it is opposite DBA and is perpendicular to BD.

(iv) BA is the *hypotenuse*.

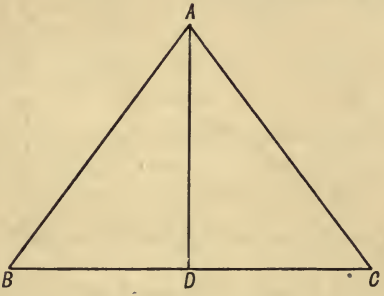
(v) BD is the *base*.

Therefore $\sin DBA$, which is $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$, $= \dfrac{DA}{BA}$,

$\cos DBA$, which is $\dfrac{\text{base}}{\text{hypotenuse}}$, $= \dfrac{BD}{BA}$.

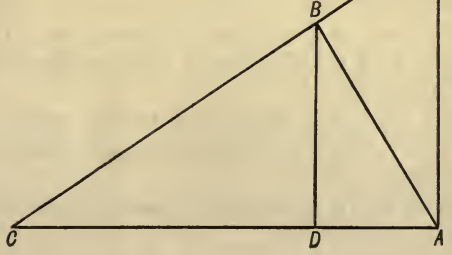
EXAMPLES. IV.

1. Let ABC be any triangle and let AD be drawn perpendicular to BC. Write down the *perpendicular*, and the *base* when the following angles are referred to: (i) the angle ABD, (ii) the angle BAD, (iii) the angle ACD, (iv) the angle DAC.



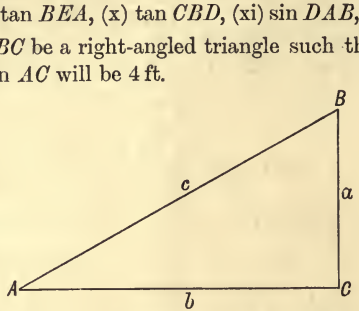
2. Write down the following ratios in the above figure; (i) $\sin BAD$, (ii) $\cos ACD$, (iii) $\tan DAC$, (iv) $\sin ABD$, (v) $\tan BAD$, (vi) $\sin DAC$, (vii) $\cos DCA$, (viii) $\tan DCA$, (ix) $\cos ABD$, (x) $\sin ACD$.

3. Let ACB be any angle and let ABC and BDC be right angles; (see next figure). Write down *two* values for each of the following ratios; (i) $\sin ACB$, (ii) $\cos ACB$, (iii) $\tan ACB$, (iv) $\sin BAC$, (v) $\cos BAC$, (vi) $\tan BAC$.



4. In the figure on the last page, BDC, CBA and EAC are right angles. Write down (i) sin DBA, (ii) sin BEA, (iii) sin CBD, (iv) cos BAE, (v) cos BAD, (vi) cos CBD, (vii) tan BCD, (viii) tan DBA, (ix) tan BEA, (x) tan CBD, (xi) sin DAB, (xii) sin BAE.

5. Let ABC be a right-angled triangle such that $AB=5$ ft., $BC=3$ ft., then AC will be 4 ft.



Find the sine, cosine and tangent of the angles at A and B, respectively.

In the above triangle if A stand for the angle at A and B for the angle at B, show that $\sin^2 A+\cos^2 A=1$, and that $\sin^2 B+\cos^2 B=1$.

6. If ABC be any right-angled triangle with a right angle at C, and let A, B, and C stand for the angles at A, B and C respectively, and let a, b and c be the measures of the sides opposite the angles A, B and C respectively.

Show that $\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$, $\tan A=\frac{a}{b}$.

Show also that $\sin^2 A+\cos^2 A=1$.

Show also that (i) $a=c.\sin A$, (ii) $b=c.\sin B$, (iii) $a=c.\cos B$, (iv) $b=c.\cos A$, (v) $\sin A=\cos B$, (vi) $\cos A=\sin B$, (vii) $\tan A=\cot B$.

7. The sides of a right-angled triangle are in the ratio 5 : 12 : 13. Find the sine, cosine and tangent of each acute angle of the triangle.

8. The sides of a right-angled triangle are in the ratio $1 : 2 : \sqrt{3}$. Find the sine, cosine and tangent of each acute angle of the triangle.

9. Prove that if A be either of the angles of the above two triangles $\sin^2 A+\cos^2 A=1$.

CHAPTER III.

On the Trigonometrical Ratios of Certain Angles.

27. The Trigonometrical Ratios of an angle are *numerical quantities simply*, as their name ratio implies. They are in nearly all cases incommensurable numbers.

Their practical value has been found for all angles between 0 and 90°, which differ by 1′; and a list of these values will be found in any volume of Mathematical Tables.

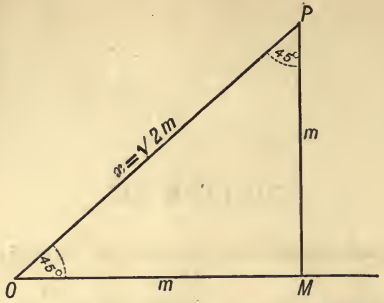
The student is recommended to get a copy of Chamber's Mathematical Tables for instruction and reference.

28. The finding the values of these Ratios has involved a large amount of labour; but, as the results have been published in Tables, the finding the Trigonometrical Ratios does not form any part of a student's work, except to exemplify the method employed.

29. The general method of finding Trigonometrical Ratios belongs to a more advanced part of the subject than the present, but there are certain angles whose Ratios can be found in a simple manner.

30. *To find the sine, cosine and tangent of the angle* 45°.

If one angle of a right-angled triangle be 45°, that is, the half of a right angle, the third angle must also be 45°. Hence 45° is one angle of an isosceles right-angled triangle.



Let POM be an isosceles triangle such that PMO is a right angle, and $OM = MP$. Then $POM = OPM = 45°$.

Let the measures of OM and of MP each be m. Let the measure of OP be x.

Then $$x^2 = m^2 + m^2 = 2m^2;$$

$$\therefore\ x = \sqrt{2}\,.\,m.$$

Hence, $$\sin 45° = \sin POM = \frac{MP}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$

$$\cos 45° = \cos POM = \frac{OM}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$

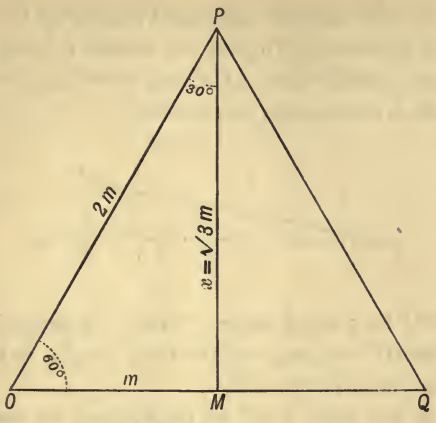
$$\tan 45° = \tan POM = \frac{MP}{OM} = \frac{m}{m} = \frac{1}{1} = 1.$$

31. *To find the sine, cosine and tangent of* 60°.

In an equilateral triangle, each of the equal angles is 60°, because they are each one third of 180°. And if we draw a perpendicular from one of the angular points of the triangle to the opposite side, we get a right-angled triangle in which one angle is 60°.

Let OPQ be an equilateral triangle. Draw PM perpendicular to OQ. Then OQ is bisected in M.

Let the measure of OM be m; then that of OQ is $2m$ and therefore that of OP is $2m$.



Let the measure of MP be x.

Then $$x^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2,$$

$$\therefore\ x = \sqrt{3}\,.\,m.$$

Hence, $$\sin 60^\circ = \sin POM = \frac{MP}{OP} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$

$$\cos 60^\circ = \cos POM = \frac{OM}{OP} = \frac{m}{2m} = \frac{1}{2},$$

$$\tan 60^\circ = \tan POM = \frac{MP}{OM} = \frac{\sqrt{3}\,.\,m}{m} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

32. *To find the sine, cosine and tangent of* 30°.

With the same figure and construction as above, we have the angle $OPM = 30^\circ$, since it is a half of OPQ, i.e. of 60°

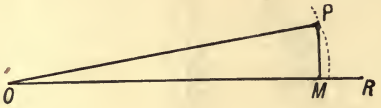
Hence, $$\sin 30^\circ = \sin OPM = \frac{MO}{PO} = \frac{m}{2m} = \frac{1}{2},$$

$$\cos 30^\circ = \cos OPM = \frac{PM}{PO} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$

$$\tan 30^\circ = \tan OPM = \frac{MO}{PM} = \frac{m}{\sqrt{3}\,.\,m} = \frac{1}{\sqrt{3}}.$$

33. *To find the sine, cosine and tangent of* 0°.

By this is meant,—*To find the values, if any, to which the Trigonometrical Ratios of a very small angle approach, as the angle is continually diminished.*



Let *ROP* be a small angle. Draw *PM* perpendicular to *OR*, and let *OP* be always of the same length, so that *P* lies on a circle whose centre is *O*.

Then if the angle *ROP* be diminished, we can see that *MP* is diminished also, and that consequently $\frac{MP}{OP}$, which is sin *ROP*, is diminished. And, by diminishing the angle *ROP* sufficiently, *we can make MP as small as we please*, and therefore *we can make* sin *ROP smaller than any assignable number however small that number may be.*

Thus we see that the value to which sin *ROP* approaches as the angle is diminished, is 0.

This is expressed by saying, sin 0° = 0.....i.

Again, as the angle *ROP* diminishes, *OM* approaches *OP* in length; and cos *ROP*, which is $\frac{OM}{OP}$, approaches in value to $\frac{OP}{OP}$, i.e. to 1.

This is expressed by saying, cos 0° = 1 .....ii.

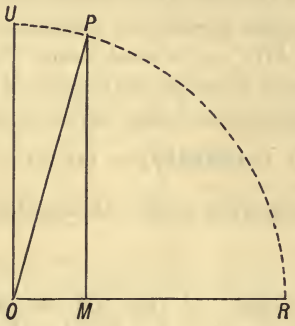
Also, tan *ROP* is $\frac{MP}{OM}$; and we have seen that *MP* approaches 0, while *OM* does not; ∴ tan *ROP* approaches 0.

This is expressed by saying, tan 0° = 0.....iii.

34. *To find the sine, cosine and tangent of* 90°.

By this is meant,—*To find the values, if any, to which the Trigonometrical Ratios of an angle approach, as the angle approaches a right angle.*

Let ROU be a right angle = 90°.



Draw ROP nearly a right angle; draw PM perpendicular to OR, and let OP be always of the same length, so that P lies on a circle whose centre is O.

Then, as the angle ROP approaches to ROU, we can see that MP approaches OP, while OM continually diminishes.

Hence when ROP approaches 90°, sin ROP, which is $\frac{MP}{OP}$, approaches in value to $\frac{OP}{OP}$, that is to $\frac{1}{1}$, i.e. to 1.

Hence we say, that sin 90° = 1 .....i.

Again, when ROP approaches 90°, cos ROP, which is $\frac{OM}{OP}$, approaches in value to $\frac{0}{OP}$, that is to 0.

Hence we say, that cos 90° = 0 .....ii.

Again, when ROP approaches 90°, tan ROP which is $\frac{MP}{OM}$ approaches in value to $\frac{OP}{\text{a quantity which approaches } 0}$.

But in any fraction whose numerator does not diminish, the smaller the denominator the greater is the value of that fraction; and if the denominator continually diminishes the value of the fraction continually increases.

Hence, *tan ROP can be made larger than any assignable number by making the angle ROP approach* 90° *near enough.*

This is what we mean when we say, that

tan 90° is **infinity,** or, tan 90° = ∞ .....iii.

35. The following table exhibits the results of this Chapter.

angle	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

The student may notice that the sine increases with the angle, while the cosine diminishes as the angle increases.

Also that the squares of the sines of 0°, 30°, 45°, 60° and 90° are respectively 0, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, and $\frac{4}{4}$, and that the squares of the cosines of the same angles are $\frac{4}{4}$, $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$, and 0.

EXAMPLES. V.

If $A=90°$, $B=60°$, $C=30°$, $D=45°$, $E=0$, prove the following:

1. $2 . \sin D . \cos D = \sin A$.
2. $2 . \sin C . \cos C = \sin B$.
3. $\cos^2 B - \sin^2 B = 1 - 2\sin^2 B$.
4. $\sin B . \cos C + \sin C . \cos B = \sin A$.
5. $\cos^2 D - \sin^2 D = \cos A$.
6. $\cos^2 E + 2 . \sin E = 1$.
7. $\sin^2 B + \cos^2 B = 1$.
8. $\cos^2 C + \sin^2 C = 1$.
9. $\cos^2 D + \sin^2 D = 1$.
10. $\sin B . \cos C - \sin C . \cos B = \sin C$.
11. $2(\cos B . \cos D + \sin B . \sin D)^2 = 1 + \cos C$.
12. $2(\sin D . \cos C - \sin C . \cos D)^2 = 1 - \cos C$.
13. $\sin 30° = \cdot 5$.
14. $\sin 45° = \cdot 7071....$
15. $\sin 60° = \cdot 8660....$
16. $\tan 60° = 1\cdot 7320508....$
17. $\tan 30° = \cdot 5773....$
18. $\sin 18° = \cdot 3090....$

36. The actual measurement of the *line* joining two points which are any considerable distance apart, is a very tedious and difficult operation, especially when great accuracy is required; while the accurate measurement of an *angle* can, with proper instruments, be made with comparative ease and quickness.

37. A **Sextant** is an instrument for measuring the angle between the two lines drawn from the observer's eye to each of two distant objects respectively.

A **Theodolite** is an instrument for measuring angles in a horizontal plane; also for measuring '*angles of elevation*' and '*angles of depression.*'

38. The angle made with the horizontal plane, by the line joining the observer's eye with a distant object, is called

(i) its **angle of elevation**, when the object is *above* the observer;

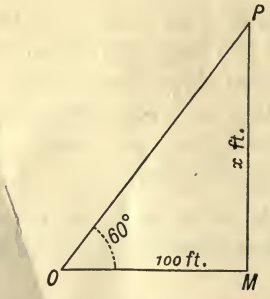
(ii) its **angle of depression**, when the object is *below* the observer. †

39. **Trigonometry** enables us by measuring certain *angles*, to deduce, from one known distance, the *lengths* of other distances: or, by the measurement of a *convenient* line, to deduce by the measurement of *angles* the lengths of lines whose actual measurement is difficult or impossible.

[In the Trigonometrical Survey of England, made by the Ordnance Department, the only distance actually measured was one of about seven miles on Salisbury Plain.]

40. For this purpose we require the numerical values of the Trigonometrical Ratios of the angles observed. Accordingly mathematical tables have been compiled, containing *lists* of the values of these Ratios. These Tables constitute a kind of numerical **Dictionary**, in which we can find the numerical value of the Trigonometrical Ratios of any required angle.

Example. At a point 100 feet from the foot of a tower the angle of elevation of the top of the tower is observed to be 60°. Find the height above the point of observation of the top of the tower.



† In measuring the angle of *depression* the telescope is turned from a horizontal position *downwards*. See Ex. 3, p. 25.

Let O be the point of observation; let P be the top of the tower; let a horizontal line through O meet the foot of the tower at the point M. Then $OM=100$ feet, and the angle $MOP=60°$. Let MP contain x feet.

Then $$\frac{MP}{OM}=\tan MOP=\tan 60°=\sqrt{3}.$$

$$\therefore \frac{x}{100}=\sqrt{3}.$$

$$\therefore x=100.\sqrt{3}=100\times 1{\cdot}7320 \text{ etc.}$$

$$=173{\cdot}2.$$

Therefore the required height is 173·2 feet.

EXAMPLES. VI.

The answers are given correct to three significant figures.

1. At a point 179 feet in a horizontal line from the foot of a column, the angle of elevation of the top of the column is observed to be 45°. What is the height of the column?

2. At a point 200 feet from, and on a level with, the base of a tower, the angle of elevation of the top of the tower is observed to be 60°: what is the height of the tower?

3. From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30°: find the height of the cliff.

4. From the top of a tower 117 feet high the angle of depression of the top of a house 37 feet high is observed to be 30°: how far is the top of the house from the tower?

5. From a point P on the bank of a river, just opposite a post Q on the other bank, a man walks at right angles to PQ to a point R so that PR is 100 yards; he then observes the angle PRQ to be 32° 17′: find the breadth of the river.

$$\tan 32° 17' = {\cdot}6317667.$$

6. A flagstaff 25 feet high stands on the top of a house; from a point on the plain on which the house stands the angles of elevation of the top and bottom of the flagstaff are observed to be 60° and 45° respectively: find the height of the house above the point of observation.

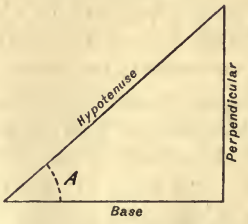
CHAPTER IV.

On the Relations between the Trigonometrical Ratios of the Same Angle.

41. The following relations are evident from the definitions:

$$\operatorname{cosec} A = \frac{1}{\sin A}, \qquad \sec A = \frac{1}{\cos A}, \qquad \cot A = \frac{1}{\tan A}.$$

42. To prove $\tan A = \dfrac{\sin A}{\cos A}$.



We have $\sin A = \dfrac{\text{perpendicular}}{\text{hypotenuse}}$,

and $\cos A = \dfrac{\text{base}}{\text{hypotenuse}}$;

$$\therefore \frac{\sin A}{\cos A} = \frac{\text{perpendicular}}{\text{base}} = \tan A.$$

43. We may prove similarly $\cot A = \dfrac{\cos A}{\sin A}$.

Or thus, $\cot A = \dfrac{1}{\tan A} = \dfrac{\cos A}{\sin A}$.

44. *To prove that* $\cos^2 \mathbf{A} + \sin^2 \mathbf{A} = 1$.

Let ROE be any angle A.

In OE take any point P, and draw PM perpendicular to OR. Then with respect to A, MP is the perpendicular, OP is the hypotenuse, and OM is the base;

Fig. I.

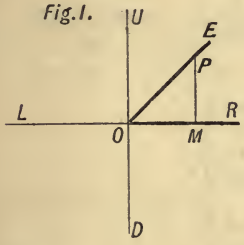


Fig. II.

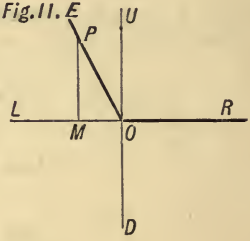


Fig. III.

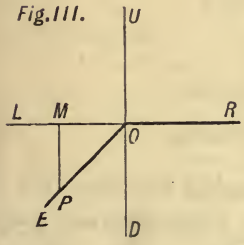
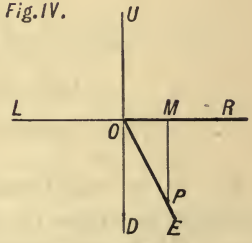


Fig. IV.



$$\therefore \sin^2 A = \frac{MP^2}{OP^2}, \qquad \cos^2 A = \frac{OM^2}{OP^2},$$

and by Euclid I. 47, $MP^2 + OM^2 = OP^2$, therefore

$$\cos^2 A + \sin^2 A = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{MP^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1. \quad \text{Q.E.D.}$$

45. Similarly we may prove that

$$1 + \tan^2 A = \sec^2 A,$$

and that

$$1 + \cot^2 A = \operatorname{cosec}^2 A.$$

NOTE. The three results $\left\{ \begin{array}{ll} \text{(i)} & \cos^2 A + \sin^2 A = 1 \\ \text{(ii)} & 1 + \tan^2 A = \sec^2 A \\ \text{(iii)} & 1 + \cot^2 A = \operatorname{cosec}^2 A \end{array} \right\}$,

are each a statement in Trigonometrical language of Euc. I. 47.

46. The following is a LIST OF FORMULÆ with which the student must make himself familiar:

$$\operatorname{cosec} A = \frac{1}{\sin A},$$

$$\sec A = \frac{1}{\cos A},$$

$$\cot A = \frac{1}{\tan A},$$

$$\tan A = \frac{\sin A}{\cos A},$$

$$\cot A = \frac{\cos A}{\sin A},$$

$$\sin^2 A + \cos^2 A = 1,$$

$$\tan^2 A + 1 = \sec^2 A,$$

$$\cot^2 A + 1 = \operatorname{cosec}^2 A.$$

47. In proving Trigonometrical identities it is often convenient to express the other Trigonometrical Ratios *in terms of the sine and cosine.*

Example. Prove that tan A + cot A = *sec* A . *cosec* A.

Since $\tan A = \frac{\sin A}{\cos A}$, $\cot A = \frac{\cos A}{\sin A}$,

$$\sec A = \frac{1}{\cos A}, \quad \operatorname{cosec} A = \frac{1}{\sin A},$$

Therefore,

$$\tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \,.\, \sin A} = \frac{1}{\cos A \,.\, \sin A}.$$

Q. E. D

48. Sometimes it is more convenient to express all the other Trigonometrical Ratios in terms of the *sine* only, or in terms of the *cosine* only.

Example (i). *Prove that*

$$\sin^4\theta + 2\sin^2\theta\cos^2\theta = 1 - \cos^4\theta.$$

Here the right-hand expression does not contain $\sin\theta$; hence for $\sin^2\theta$ in the left-hand expression we substitute $(1-\cos^2\theta)$; thus

$$\begin{aligned}\sin^4\theta + 2\sin^2\theta\cos^2\theta &= (\sin^2\theta)^2 + 2(\sin^2\theta)\cos^2\theta \\ &= (1-\cos^2\theta)^2 + 2(1-\cos^2\theta)\cos^2\theta \\ &= 1 - 2\cos^2\theta + \cos^4\theta + 2\cos^2\theta - 2\cos^4\theta \\ &= 1 - \cos^4\theta. \quad \text{Q.E.D.}\end{aligned}$$

Example (ii). *Prove that*

$$\sin^6\theta + \cos^6\theta = 1 - 3\cos^2\theta + 3\cos^4\theta.$$

Here

$$\begin{aligned}\sin^6\theta + \cos^6\theta &= (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) \\ &= 1 \times (\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) \\ &= (\sin^2\theta)^2 - (\sin^2\theta)\cos^2\theta + \cos^4\theta \\ &= (1-\cos^2\theta)^2 - (1-\cos^2\theta)\cos^2\theta + \cos^4\theta \\ &= 1 - 2\cos^2\theta + \cos^4\theta - \cos^2\theta + \cos^4\theta + \cos^4\theta \\ &= 1 - 3\cos^2\theta + 3\cos^4\theta. \quad \text{Q.E.D.}\end{aligned}$$

NOTE. $(1-\cos\theta)$ is called the **versed sine** of θ; it is abbreviated thus versin θ.

EXAMPLES. VII.

Prove the following statements.

1. $\cos A \,.\, \tan A = \sin A.$
2. $\cot A \,.\, \tan A = 1.$
3. $\cos A = \sin A \,.\, \cot A.$
4. $\sec A \,.\, \cot A = \operatorname{cosec} A.$
5. $\operatorname{cosec} A \,.\, \tan A = \sec A.$
6. $(\tan A + \cot A)\sin A \,.\, \cos A = 1.$
7. $(\tan A - \cot A)\sin A \,.\, \cos A = \sin^2 A - \cos^2 A.$

8. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A.$

9. $(\sin A + \cos A)^2 = 1 + 2\sin A \,.\, \cos A.$

10. $(\sin A - \cos A)^2 = 1 - 2\sin A \,.\, \cos A.$

11. $\cos^4 B - \sin^4 B = 2\cos^2 B - 1.$

12. $(\sin^2 B + \cos^2 B)^2 = 1.$

13. $(\sin^2 B - \cos^2 B)^2 = 1 - 4\cos^2 B + 4\cos^4 B.$

14. $1 - \tan^4 B = 2\sec^2 B - \sec^4 B.$

15. $(\sec B - \tan B)(\sec B + \tan B) = 1.$

16. $(\operatorname{cosec}\theta - \cot\theta)(\operatorname{cosec}\theta + \cot\theta) = 1.$

17. $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta).$

18. $\cos^3\theta - \sin^3\theta = (\cos\theta - \sin\theta)(1 + \sin\theta\cos\theta).$

19. $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta \,.\, \cos^2\theta.$

20. $(\sin^6\theta - \cos^6\theta) = (2\sin^2\theta - 1)(1 - \sin^2\theta + \sin^4\theta).$

21. $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \,.\, \tan B.$

22. $\dfrac{\cot a + \tan\beta}{\tan a + \cot\beta} = \cot a \,.\, \tan\beta.$

23. $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2.$

24. $\dfrac{1 + \cos A}{1 - \cos A} = (\operatorname{cosec} A + \cot A)^2.$

25. $2\operatorname{versin}\theta - \operatorname{versin}^2\theta = \sin^2\theta.$

26. $\operatorname{versin}\theta(1 + \cos\theta) = \sin^2\theta.$

27. $\dfrac{\tan A + \sec A + 1}{\tan A + \sec A - 1} = \dfrac{\sec A + 1}{\tan A}.$

28. $4(\cos^6 A + \sin^6 A) - 3(\cos^4 A - \sin^4 A)^2 = 1.$

29. $(\sec^2 A + \tan^2 A)(\operatorname{cosec}^2 A + \cot^2 A) = 1 + 2\sec^2 A \operatorname{cosec}^2 A.$

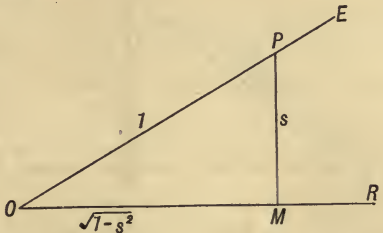
30. $\dfrac{1}{\tan^2 A + 1} + \dfrac{1}{\cot^2 A + 1} = 1.$

31. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A.$

32. $\cot^4 A + \cot^2 A = \operatorname{cosec}^4 A - \operatorname{cosec}^2 A.$

49. All the Trigonometrical Ratios of an angle can be expressed in terms of any one of them.

50. *To express all the trigonometrical ratios of an angle in terms of the sine.*



Let ROE be any angle A.

We can take P anywhere in the line OE; so that we can make *one* of the lines, OP, OM, or MP any length we please. Let us take OP so that its measure is 1. Let s be the measure of MP; so that $\sin A$, which is $\frac{MP}{OP}$, $= \frac{s}{1}$; or, $s = \sin A$. Let x be the measure of OM.

Then since $OM^2 = OP^2 - MP^2$,

$$\therefore\ x^2 = 1 - s^2,$$

$$\therefore\ x = \sqrt{1 - s^2}.$$

Hence $$\cos A = \frac{OM}{OP} = \frac{\sqrt{1-s^2}}{1} = \sqrt{1 - \sin^2 A},$$

$$\tan A = \frac{MP}{OM} = \frac{s}{\sqrt{1-s^2}} = \frac{\sin A}{\sqrt{1 - \sin^2 A}},$$

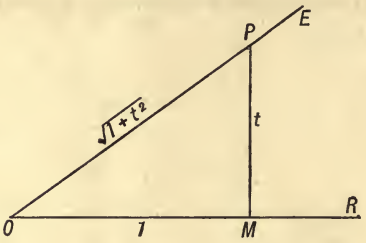
and so on.

NOTE. The solution of the equation $x^2 = 1 - s^2$, gives

$$\sqrt{1-s^2},$$

and therefore the ambiguity ($\pm$) must stand before each of the root symbols in the above. This ambiguity, as will be explained later on, is necessary when the magnitude of the angle is not limited. When we limit A to be less than a right angle we have no use for the negative sign.

51. *To express all the other trigonometrical ratios of an angle in terms of the tangent.*



In this case $\tan\theta = \frac{MP}{OM}$.

Take P so that the measure of OM is 1, and let t be the measure of MP; so that $\tan\theta$, which is $\frac{MP}{OM}$, $= \frac{t}{1}$; or, $t = \tan\theta$.

Then we can show that the measure of OP is $\sqrt{1+t^2}$.

Hence, $$\sin\theta = \frac{MP}{OP} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}},$$

$$\cos\theta = \frac{OM}{OP} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\theta}},$$

and so on.

52. The same results may be obtained by the use of the formulæ on page 28.

Example. $\cos^2\theta + \sin^2\theta = 1,$

$\therefore \cos^2\theta = 1 - \sin^2\theta,$

$\therefore \cos\theta = \sqrt{1-\sin^2\theta}.$

Again $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}},$$

and so on.

EXAMPLES. VIII.

1. Express all the other Trigonometrical Ratios of A in terms of cos A.

2. Express all the other Trigonometrical Ratios of A in terms of cot A.

3. Express all the other Trigonometrical Ratios of A in terms of sec A.

4. Express all the other Trigonometrical Ratios of A in terms of cosec A.

5. Use the formulæ of page 28 to express all the other Trigonometrical Ratios of A in terms of sin A.

6. Use the formulæ of page 28 to express all the other Trigonometrical Ratios of A in terms of tan A.

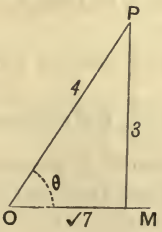
53. *Given one of the Trigonometrical Ratios of an angle less than a right angle, we can find all the others.*

Since all the Trigonometrical Ratios of an angle can be expressed in terms of any one of them, it is clear that if the numerical value of any one of them be given, the numerical value of all the rest can be found.

Example. Given $\sin\theta = \frac{3}{4}$, find the other Trigonometrical Ratios of θ.

Let ROE be the angle θ.

Take P on OE so that the measure of OP is 4. Draw PM perpendicular to OR.



Then since $\sin\theta=\frac{3}{4}$ $\left(\text{so that } \frac{MP}{OP}=\frac{3}{4}\right)$, and since the measure of OP is 4, therefore the measure of MP must be 3.

Let x be the measure of OM;

then $$OM^2=OP^2-MP^2,$$

$$\therefore\ x^2=4^2-3^2=16-9=7.$$

$$x=\sqrt{7}.$$

Therefore the measure of OM is $\sqrt{7}$.

Hence, $$\cos\theta=\frac{OM}{OP}=\frac{\sqrt{7}}{4},$$

$$\tan\theta=\frac{MP}{OM}=\frac{3}{\sqrt{7}}=\frac{3\sqrt{7}}{7},$$

$$\cot\theta=\frac{\sqrt{7}}{3}.$$

EXAMPLES. IX.

1. Given that $\sin A=\frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.
2. Given that $\cos B=\frac{1}{3}$, find $\sin B$ and $\cot B$.
3. Given that $\tan A=\frac{4}{3}$, find $\sin A$ and $\sec A$.
4. Given that $\sec\theta=4$, find $\cot\theta$ and $\sin\theta$.
5. Given that $\tan\theta=\sqrt{3}$, find $\sin\theta$ and $\cos\theta$.
6. Given that $\cot\theta=\frac{2}{\sqrt{5}}$, find $\sin\theta$ and $\sec\theta$.
7. Given that $\sin\theta=\frac{b}{c}$, find $\tan\theta$.
8. Given that $\tan\theta=\frac{a}{b}$, find $\sin\theta$ and $\cos\theta$.
9. Given that $\cos\theta=\frac{1}{a}$, find $\sin\theta$ and $\cot\theta$.
10. If $\sin\theta=a$, and $\tan\theta=b$, prove that $(1-a^2)(1+b^2)=1$.
11. If $\cos\theta=h$, and $\tan\theta=k$, find the equation connecting h and k.

54. Since the hypotenuse is the *greatest* side in a right-angled triangle, it is clear

(i) that **sin A**, which is $\frac{\text{perpendicular}}{\text{hypotenuse}}$, **is never greater than unity,**

(ii) that **cos A**, which is $\frac{\text{base}}{\text{hypotenuse}}$, **is never greater than unity,**

(iii) that **cosec A**, which is $\frac{\text{hypotenuse}}{\text{perpendicular}}$, **is never numerically less than unity** (it may be *negative*),

(iv) that **sec A**, which is $\frac{\text{hypotenuse}}{\text{base}}$, **is never numerically less than unity** (it may be negative).

Hence, such a question as '*Find an angle whose* **sine** *is* $\frac{4}{3}$' has no solution, since there is no angle whose sine is greater than unity.

And such a question as '*Find an angle whose* **secant** *is* $\frac{3}{4}$' has no solution, because there is no angle whose secant is numerically less than unity.

55. If A be small, the perpendicular is smaller than the base; and tan A, which is $\frac{\text{perpendicular}}{\text{base}}$, can be made as small as we please (see figure in Art. 33). Also if the angle A be nearly a right angle, the perpendicular is greater than the base; and tan A, which is $\frac{\text{perpendicular}}{\text{base}}$, can be made as large as we please.

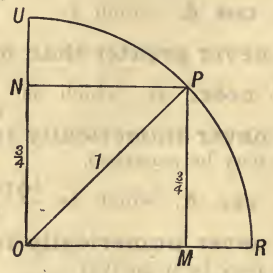
So that an angle whose **tangent** is a can *always* be found if a be positive; that is, there is always an angle A between 0° and 90°, such that tan $A = a$, if a be a positive number.

[The student will find as he proceeds that the equation $\tan x = a$ can always be solved if a be any *arithmetical* quantity.]

Example 1. *Draw an angle* A *whose sine is* $\frac{3}{4}$.

Draw any line *OR*, and take *R* such that the measure of *OR* is unity; describe the quadrant *RPU* of the circle whose centre is *O* and radius *OR*.

In *OU* which is perpendicular to *OR*, take *ON* so that the measure of *ON* is $\frac{3}{4}$.



Draw *NP* parallel to *OR* cutting the quadrant in *P*. (The student should observe that if the measure of *ON* were *greater than unity*, *N* would be outside the circle altogether, and the line parallel to *OR* would not cut the circle at all.)

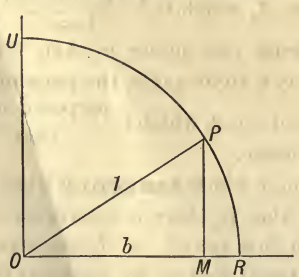
Join *OP*, and draw *PM* perpendicular to *OR*.

Then *ROP* is the angle required.

For $\sin ROP = \frac{MP}{OP} = \frac{ON}{OP} = \frac{3}{4} \div 1 = \frac{3}{4}$.

Therefore an angle *POR* has been drawn whose sine is $\frac{3}{4}$.

Example 2. *Draw an angle whose cosine is* b; where *b* is a *proper* fraction.



Draw any line *OR*, and take *OR* so that its measure is unity.

Describe the quadrant *RPU* of the circle whose centre is *O* and radius *OR*.

Take M in OR so that the measure of OM is b. OM will be less than OR because b is a *proper* fraction.

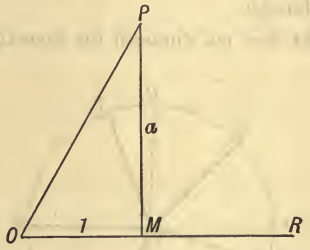
Draw MP perpendicular to OR cutting the quadrant in P. (The student will observe that if b were an *improper* fraction, M would be outside the circle altogether.)

Then the angle ROP is the angle required.

For $\cos ROP = \frac{OM}{OP} = \frac{b}{1} = b$.

Therefore an angle ROP has been described whose cosine is b.

Example 3. *Draw an angle A such that* $\tan A = a$, where a is any positive numerical quantity.



Draw any line OR, and take M in it so that the measure of OM is unity.

Draw MP perpendicular to OM, and take P so that the measure of MP is a. Join OP. Then the angle ROP is the angle required.

For $\tan ROP = \frac{MP}{OM} = \frac{a}{1} = a$.

Thus an angle ROP has been drawn so that $\tan ROP = a$.

EXAMPLES. X.

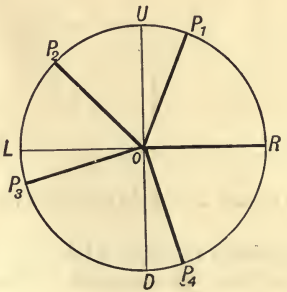
1. Draw an angle whose sine is $\frac{1}{2}$.
2. Draw an angle whose cosecant is 2.
3. Draw an angle whose tangent is 2.
4. Can an angle be drawn whose tangent is 431?
5. Can an angle be drawn whose cosine is $\frac{4}{3}$?
6. Can an angle be drawn whose secant is 5?

CHAPTER V.

Numerical Values of Angles greater than 90°.

56. It is often convenient to keep the revolving line of a particular length.

This by Art. 24 does not diminish the generality of our conclusions.



In this case the point P lies always on the circumference of a circle whose centre is O.

Let this circle cut the lines LOR, UOD in the points L, R, U, D respectively.

The circle $RULD$ is thus divided at the points R, U, L, D into four **Quadrants**, of which

RU is called the **first Quadrant.**
UL is called the **second Quadrant.**
LD is called the **third Quadrant.**
DR is called the **fourth Quadrant.**

Hence, in the figure,

ROP_1 is an angle of the first Quadrant.
ROP_2 „ „ second Quadrant.
ROP_3 „ „ third Quadrant.
ROP_4 „ „ fourth Quadrant.

57. When we speak of an angle of a particular quadrant say the second, it does not follow that the angle lies between 90° and 180°. The revolving line may have performed a number of complete revolutions positive or negative before it stopped in the second quadrant

58. Let $A°$ be an angle between 0° and 90°, and let n be any *whole* number, positive or negative.

Then

i $2n \times 180° + A°$ represents an angle of the *first* Quadrant.

ii $2n \times 180° - A°$ represents an angle of the *fourth* Quadrant.

[For $2n \times 180°$ represents some number n of *complete* revolutions of OP; so that after describing $n \times 360°$, OP is again in the position OR.]

iii $(2n+1) \times 180° - A°$ represents an angle of the second quadrant.

iv $(2n+1) \times 180° + A°$ represents an angle of the third quadrant.

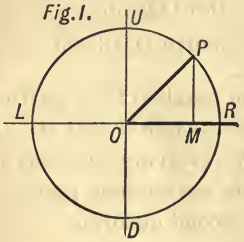
EXAMPLES. XI.

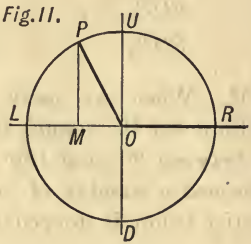
State in which Quadrant the revolving line will be after describing the following angles :

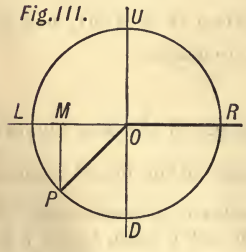
1. 120°. **2.** 340°. **3.** 490°. **4.** $-100°$.

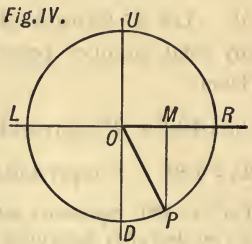
5. $-380°$. **6.** $-1000°$. **7.** $n \times 360° - 500°$. **8.** $n \times 360° + 400°$.

59. We proceed to examine more particularly the changes in the Trigonometrical Ratios according to the Quadrant to which the angle belongs.









We observe

That *MP* is *positive* in the *first* and in the *second* Quadrant.

Because the direction of *MP* is then from *O* to *U*.

That *MP* is negative in the *third* and *fourth* Quadrants.

That *OM* is *positive* in the *first* and *fourth* Quadrants.

Because the direction of *OM* is then from *O* to *R*.

That *OM* is *negative* in the *second* and *third* Quadrants.

That *OP* does not undergo any change of direction except that due to the tracing of the angle and is therefore *always positive.*

Now $\sin A = \frac{MP}{OP}$, $\cos A = \frac{OM}{OP}$, $\tan A = \frac{MP}{OM}$.

Hence. In the **First** Quadrant *MP*, *OM*, and *OP* are each positive and therefore the trigonometrical ratios are each *positive*.

In the **Second** Quadrant,

sin *ROP* is *positive*; cos *ROP* is *negative*; tan *ROP* is *negative*.

In the **Third** Quadrant,

sin *ROP* is *negative*; cos *ROP* is *negative*; tan *ROP* is *positive*.

In the **Fourth** Quadrant,

sin *ROP* is *negative*; cos *ROP* is *positive*; tan *ROP* is *negative*.

60. The table given below exhibits the results of the last article.

Quadrant ...	I.	II.	III.	IV.
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent ...	+	−	+	−

The student should notice that in any particular Quadrant the three *signs* of sine, cosine, and tangent are unlike their signs in any other Quadrant.

EXAMPLES. XII.

State the *sign* of the sine, cosine, and tangent of each of the following angles:

1. 60°.	**2.** 135°.	**3.** 265°.
4. 275°.	**5.** −10°.	**6.** −91°.
7. −193°.	**8.** −350°.	**9.** −1000°.
10. $2n\,180° + 45°$.	**11.** $2n\,180° + 135°$.	**12.** $2n\,180° - 30°$.

61. The NUMERICAL VALUES through which the Trigonometrical Ratios pass, as the angle ROP moves through the *first* Quadrant, are **repeated** as the angle ROP moves through each of the other Quadrants.

Thus as P moves through the second Quadrant from U to L, Fig. II. (OP being always of the same length) MP and OM pass through the same succession of numerical values through which they pass, as P moves through the first Quadrant in the *opposite* direction from U to R.

Example 1. *Find the sine, cosine and tangent of* 120°.

120° is an angle in the second Quadrant.

Let the angle ROP be 120° (Fig. II. p. 40).

Then the angle $POL = 180° - 120° = 60°$.

Hence, $\sin 120° = \frac{MP}{OP} = \sin 60°$ *numerically*, and in the *second* Quadrant the sine is *positive*.

Therefore $\sin 120° = \frac{\sqrt{3}}{2}$ .....(i).

Again, $\cos 120° = \frac{OM}{OP} = \cos 60°$, *numerically*, and in the *second* Quadrant the cosine is *negative*.

Therefore $\cos 120° = -\frac{1}{2}$ .....(ii).

Similarly, $\tan 120° = -\sqrt{3}$ .....(iii).

Example 2. *Find the sine, cosine and tangent of* 225°.

225° is an angle in the third Quadrant.

Let the angle ROP be 225° (Fig. III. p. 40).

Here the angle $POL = 225° - 180° = 45°$.

Therefore the Trigonometrical Ratios of 225° = those of 45° *numerically*; and in the *third* Quadrant the sine and cosine are each *negative* and the tangent is *positive*.

Hence, $\sin 225° = -\frac{1}{\sqrt{2}}$; $\cos 225° = -\frac{1}{\sqrt{2}}$; $\tan 225° = 1$.

62. The cosecant, secant and cotangent of an angle A have the same *sign* as the sine, cosine, and tangent of A respectively.

EXAMPLES. XIII.

Find the algebraical value of the sine, cosine and tangent of the following angles :

1.	150°.	**2.**	135°.	**3.**	−240°.	**4.**	330°.
5.	−45°.	**6.**	−300°.	**7.**	225°.	**8.**	−135°.
9.	390°.	**10.**	750°.	**11.**	−840°.	**12.**	1020°.

13. $2n180°+45°$. **14.** $(2n+1)180°-60°$. **15.** $(2n-1)180°+30°$.

63. *Trace the changes in* sin A *as* A *passes through all values from* 0° *to* 360°. (See Figure on next page.)

Let the revolving line OP starting from OR revolve about O so as to trace out successively all angles below 0° and 360°.

Draw PM perpendicular to LOR and let P trace a circle.

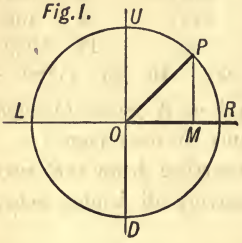
Then OP is always of the same length and we notice that MP is at first zero and then continuously increases till P arrives at U when $MP = OP$, and then again diminishes continuously till P arrives at L when MP is again zero.

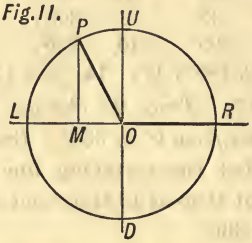
MP then becomes *negative* ; and beginning at zero when P is at L it increases continuously numerically exactly as before until P reaches D when $MP = -OP$ and then again diminishes continuously numerically being still negative until it becomes zero when P reaches R.

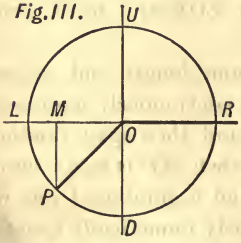
Now $\sin A = \dfrac{MP}{OP}$, and while OP continues unchanged MP goes through the above changes ; whence we see that as A increases from 0° to 360° sin A increases continuously from 0 to 1 which value it has when $A = 90°$; sin A then diminishes continuously to 0 again which value it has when $A = 180°$; it then becomes negative, and increases continuously numerically exactly as before becoming −1 when $A = 270°$ and then being still negative diminishes continuously numerically until it is again zero when A becomes 360°.

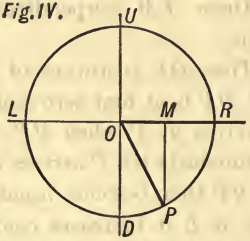
64. *Trace the changes in the sign and magnitude of* cos A *as* A *passes through all values from* 0° *to* 360°.

With construction of the last Article we notice,









That OP is always positive and of the same magnitude.

That OM is equal to OP when P is at R and as OP revolves OM diminishes continuously until it is zero when P reaches U. OM then changes sign and increases continuously numerically (exactly as MP does when OP goes from OR to OU) until it becomes equal to $-OP$ when P arrives at L. OM then diminishes continuously numerically, being still negative, until it becomes zero when P arrives at D; OM then changes sign and becomes positive and from zero increases continuously until it becomes equal to OP when P arrives again at R.

Now $\cos A = \dfrac{OM}{OP}$ and while OP is unchanged, OM goes through the above change; whence we see,

that as A increases from 0° to 360° cos A diminishes continuously from 1 to 0 which value it has when $A = 90°$.

cos A then becomes negative and increases continuously numerically from 0 to -1 which value it has when $A = 180°$; it next diminishes continuously numerically from -1 to 0 which value it has when $A = 270°$, continuing still negative; and lastly it becomes positive and increases continuously from 0 to 1 which value it has when $A = 360°$.

65. *Trace the changes in* tan A *as* A *passes continuously from* 0° *to* 360°.

With the figure and construction of Art. 64 we notice

i that MP changes continuously from 0 to 1; from 1 to 0; from 0 to -1; and from -1 to 0 as A turns continuously from 0° to 90°; from 90° to 180°; from 180° to 270°, and from 270° to 360°.

ii that OM changes continuously from 1 to 0; from 0 to -1; from -1 to 0; from 0 to 1, as A turns continuously from 0° to 90°; from 90° to 180°; from 180° to 270°; and from 270° to 360°.

Now tan $A = \dfrac{MP}{OM}$; hence we see that the fraction tan A changes continuously from 0 to ∞; from $-\infty$ to 0; from 0 to $-\infty$; and from ∞ to 0, as A turns continuously from 0° to 90°; from 90° to 180°; from 180° to 270°; and from 270° to 360°.

Here the symbol $\pm\infty$ means that as the angle A approaches its value, the tangent can be made greater than any assignable positive or negative number.

NOTE. We cannot distinguish between ∞ and $-\infty$ any more than we can between $+0$ and -0.

EXAMPLES. XIV.

Trace the changes in sign and magnitude as A increases continuously from 0° to 360° of

1. $\cos A$.	2. $\tan A$.	3. $\cot A$.	4. $\sec A$.
5. $\operatorname{cosec} A$.	6. $1-\sin A$.	7. $\sin^2 A$.	8. $\sin A \cdot \cos A$.
9. $\sin A+\cos A$.	10. $\tan A+\cot A$.		11. $\sin A-\cos A$.

66. We have said that the trigonometrical ratios of any given angle can be found from the Table (or Dictionary).

The Table gives the Ratios of angles between 0° and 90° and we have shewn how the Ratios of a given angle of any magnitude can be expressed in terms of the Ratios of an angle between 0° and 90°.

When we are given a numerical value for the sine we can always find angles having that sine provided the number is less than unity.

67. *To draw the diagram of angles having their* sine *equal to a given number* a (less than unity).

With the usual construction, let the radius of the circle $RULD$ be the unit of length; then the measure of OP is 1.

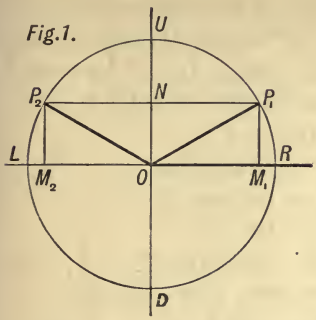
From O draw on OU a line ON so that its measure is a.

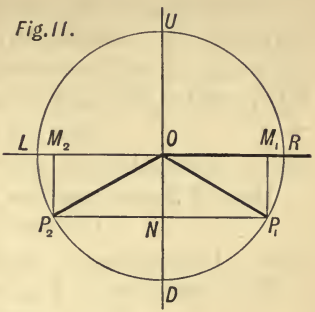
[ON will be drawn upwards (Fig. I.) or downwards (Fig. II.) from O according as a is positive or negative.]

Through N draw P_1P_2 parallel to LOR to cut the circle in P_1 and P_2*. Join OP_1, OP_2. Draw P_1M_1, P_2M_2 perpendicular to LOR.

Then $M_1P_1 = ON = M_2P_2$, and $\dfrac{M_1P_1}{OP_1} = \dfrac{a}{1} = \dfrac{M_2P_2}{OP_2}$.

* If a were *greater than unity*, ON would be greater than OU, and the construction would fail.





Hence any angle described by the revolving line OP, when OP, starting from OR, stops either in the position OP_1, or in the position OP_2, is an angle whose **sine** is a.

And no other angle has its sine equal to a, for there is no other possible position of N.

Hence the required diagram is like Fig. I. when a is positive and like Fig. II. when a is a negative.

68. *To find an algebraic expression for all angles having their sine equal to a given number* a.

With the figure and construction of the last article, let $A°$ be the least angle whose sine $= a$.

Then in either of the figures $ROP_1 = A°$ and $P_2OL = A°$.

Every angle whose sine $= a$ is included among those described by the revolving line OP when, starting from OR, OP stops in one or the other of the positions OP_1 or OP_2.

When OP, starting from OR, stops in the position OP_1, the angle may be described by OP turning through a certain number of complete revolutions to OR, and then through ROP_1; that is, the angle is one of those included in

$$2m \times 180° + A°,$$

where m is some integer, positive or negative. [Art. 58.]

When OP, starting from OR, stops in the position OP_2, the angle may be described by OP turning through a certain odd number of half-revolutions to OL, and then *back* through LOP_2; that is, the angle is one of those included in

$$(2r+1)\times 180^\circ - A^\circ,$$

where r is some integer, positive or negative.

Both of these expressions are included in

$$n\times 180^\circ + (-1)^n A^\circ,$$

where n is some integer, positive or negative *.

Thus, the solution of the equation $\sin\theta = \sin A$ is

$$\theta = n\times 180^\circ + (-1)^n A^\circ.$$

Example. Find six angles between -4 right angles and $+8$ right angles which satisfy the equation $\sin A = \sin 18^\circ$.

$$A = n\times 180^\circ + (-1)^n 18^\circ.$$

Put for n the values -2, -1, 0, 1, 2, 3, 4 successively and we get the six angles

$$-360^\circ+18^\circ,\ -180^\circ-18^\circ,\ 18^\circ,\ 180^\circ-18^\circ,\ 360^\circ+18^\circ,\ 540^\circ-18^\circ,$$

i.e. -342°, -198°, 18°, 162°, 378°, 522°.

The student is recommended to draw a figure in the above example. Also to draw a figure in each example of this kind which he works for exercise.

EXAMPLES. XV.

1. Find the four smallest angles which satisfy the equations (i) $\sin A = \frac{1}{2}$. (ii) $\sin A = \frac{1}{\sqrt{2}}$. (iii) $\sin A = \frac{\sqrt{3}}{2}$. (iv) $\sin A = -\frac{1}{2}$.

2. Find four angles between zero and $+8$ right angles which satisfy the equations

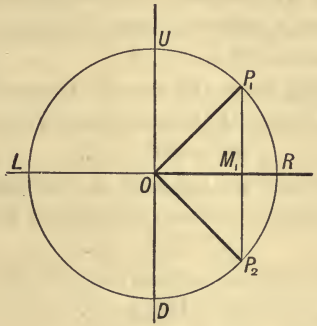
(i) $\sin A = \sin 20^\circ$. (ii) $\sin\theta = -\frac{1}{\sqrt{2}}$. (iii) $\sin\theta = -\sin\frac{\pi}{7}$.

3. Prove that 30°, 150°, -330°, 390°, -210° have all the same sine.

* For if n be *even*, let it be $2m$, when $(-1)^{2m} = +1$; if n be *odd*, let it be $2r+1$, when $(-1)^{2r+1} = -1$.

69. *To find the complete Geometrical Solution of the equation* $\cos\theta = a$.

With the usual construction, let the radius of the circle $RULD$ be the unit of length, so that the measure of OP is 1.



From O draw on OR a line OM_1 such that its measure is a.

OM_1 will be drawn towards the right or towards the left according as a is positive or negative.

Through M_1 draw $P_1M_1P_2$ perpendicular to OR to cut the circle in P_1, P_2. Join OP_1, OP_2.

Then $$\frac{OM_1}{OP_1} = \frac{a}{1} = \frac{OM_1}{OP_2}.$$

Hence any angle described when OP, starting from OR, stops in the position OP_1, or in the position OP_2, is an angle whose **cosine** is a.

And no other angle has its cosine $= a$, for there is no other possible position of M_1.

NOTE. The figure is drawn for the case in which a is positive, the second case is left to the student.

70. *To find the complete Algebraical Solution of the equation* $\cos\theta = a$.

With the figure and construction of the last article, let $A°$ be the least angle whose cosine $= a$.

Then in the figure $ROP_1 = A°$ and $P_2OR = a$.

Every angle whose cosine $= a$ is included among those described when OP, starting from OR, stops in one or the other of the positions OP_1 or OP_2.

If OP starting from OR stop in the position OP_1, the angle described is one of those included in the expression

$$2m \times 180° + A°. \qquad \text{[Art. 58.]}$$

If OP starting from OR stop in the position OP_2, the angle described is one of those included in the expression

$$2r \times 180° - A°.$$

Both of these expressions are included in

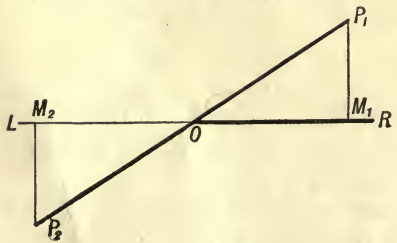
$$2n \times 180° \pm A°.$$

Thus the solution of the equation $\cos\theta = \cos a$ is

$$\theta = 2n \times 180° \pm A°.$$

71. *To find the complete Geometrical Solution of the equation* $\tan\theta = a$.

From O draw on the line OR two lines OM_1, OM_2, whose measures are $+1$ and -1 respectively.



And draw perpendicular to LOR from M_1 a line M_1P_1 whose measure is a, and from M_2 a line M_2P_2 whose measure is $-a$. Join OP_1, OP_2.

Then $\frac{M_1P_1}{OM_1} = \frac{a}{1} = a$, and $\frac{M_2P_2}{OM_2} = \frac{-a}{-1} = a$.

Hence any angle described when OP, starting from OR, stops in one or the other of the positions OP_1 or OP_2, is an angle whose **tangent** is a.

And no other angle has its tangent $= a$.

72. *To find the complete Algebraical Solution of the equation* $\tan\theta = a$.

With the figure and construction of the last article, let $A°$ be the least angle whose tangent is a.

Then in the figure $ROP_1 = A°$ and $LOP_2 = a$.

Every angle whose tangent is a is included among those described by OP, starting from OR and stopping in one or the other of the positions OP_1 or OP_2.

If OP starting from OR stop in the position OP_1, the angle described is one of those included in the expression

$$2m180° + A\,.$$ Art 58.]

If OP starting from OR stop in the position OP_2, the angle described is one of those included in the expression

$$(2r+1)\,180° + A°.$$

Both of these expressions are included in

$$n180° + A°*.$$

Thus the solution of the equation $\tan\theta = \tan a$ is

$$\theta = n180° + A°.$$

EXAMPLES. XVI.

1. Write down the complete Algebraical Solution of each of the following equations :

(i) $\cos\theta = \frac{1}{2}$. (ii) $\tan\theta = 1$. (iii) $\tan\theta = -1$.

(iv) $\tan\theta = -\sqrt{3}$. (v) $\cos\theta = \cos\frac{4\pi}{5}$. (vi) $\tan\theta = \tan\frac{3\pi}{4}$.

* For if n be even, this is the first formula ; if n be odd it is the second.

2. Shew that each of the following angles has the same cosine:

$$-120°,\ 240°,\ 480°,\ -480°.$$

3. The angles $60°$ and $-120°$ have one of the Trigonometrical Ratios the same for both; which of the ratios is it?

4. Can the following angles have any one of their Trigonometrical Ratios the same for all?

$$-23°,\ -157° \text{ and } 157°.$$

5. Find four angles which satisfy each of the equations in Question **1.**

CHAPTER VI.

The Ratios of Different Angles.

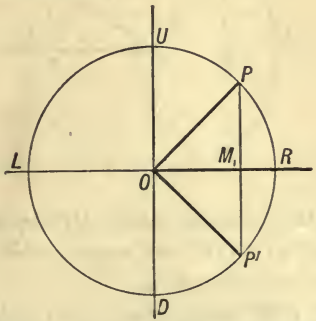
73. We proceed to consider the relations between the Trigonometrical Ratios of the angles $+A$ and $-A$, and of angles which differ by some multiple of a right angle.

74. *PROP.* *To prove* $\sin A = -\sin(-A)$,
and $\cos A = \cos(-A)$.

Take two revolving lines, OP and OP'.

Let OP starting from OR describe the angle A, let OP' starting from OR describe the angle $-A$.

So that the angle ROP is always equal geometrically to the angle ROP'; also whenever P crosses LOR, P' crosses it also; hence whenever P is *above* LOR, P' is *below* LOR and vice versa.



Let $OP = OP'$, then perpendiculars drawn from P and P' to LOR will cut LOR in the same point M, and $MP = MP'$ in magnitude.

Hence OM is common to the two angles
and MP, MP' are always equal and of opposite sign.

Hence $\frac{MP}{OP} = -\frac{MP'}{OP'}$, always;

or, $\sin A = -\sin(-A)$, for all values of A.

Also $\frac{OM}{OP} = \frac{OM}{OP'}$, always;

or, $\cos A = \cos(-A)$, for all values of A.

75. *PROP.* *To prove* $\sin A = -\cos(90° + A)$,

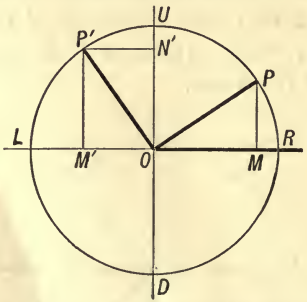
and $\cos A = \sin(90° + A)$.

Take **two revolving lines** OP and OP'.

Let OP starting from OR describe the angle A;

let OP' starting from OR describe the angle $90° + A$.

In describing $(90° + A)$ we shall consider that OP' starting from OR turns *first* through $90°$ into the position OU, and then on from OU through the angle A.



So that ROP, the angle which OP describes from OR, is always equal to UOP', the angle which OP' describes from OU in the *same* direction.

Draw PM, $P'M'$ perpendiculars to LOR; draw $P'N'$ perpendiculars to UOD.

Hence, $N'P'$, that is OM', *always* $= MP$ in *magnitude*.

Also, P' is to the left or to the right of UOD according as P is above or below LOR.

So that MP and OM' are always of *opposite sign*.

Hence, $\frac{MP}{OP} = -\frac{OM'}{OP'}$, always,

or, $\sin A = -\cos(90° + A)$, for all values of A.

Similarly, OM *always* $= M'P'$ in *magnitude.*

And P' is above or below LOR according as P is to the right or to the left of UOD.

So that OM and $M'P'$ are *always* of the *same sign.*

Hence, $\frac{OM}{OP} = \frac{M'P'}{OP'}$, always,

or, $\cos A = \sin(90° + A)$, for all values of A.

76. The above results are very important; they may be expressed as follows.

I. The sine of an angle is equal to – the sine of – the angle.

II. The cosine of an angle is equal to the cosine of – the angle.

III. The sine of the greater of two angles which differ by 90°, is equal to the cosine of the less.

IV. The cosine of the greater of two angles which differ by 90° is equal to – the sine of the less.

Example i. *Prove* $\sin(180° + A) = -\sin A$,
and $\cos(180° + A) = -\cos A$.

Here, $\sin(180° + A) = \sin(90° + \overline{90° + A}) = \cos(90° + A)$ [by III.
$= -\sin A$ [by IV.

Again, $\cos(180° + A) = \cos(90° + \overline{90° + A}) = -\sin(90° + A)$ [by IV.
$= -\cos A$ [by III.

Example ii. *Prove* $\sin(180° - A) = \sin A$,
and $\cos(180° - A) = -\cos A$.

Here, $\sin(180° - A) = \sin(90° + 90° - A) = \cos(90° - A)$ [by III.
$= -\sin(-A)$ [by IV.
$= \sin A$ [by I.

Again, $\cos(180° - A) = \cos(90° + 90° - A) = \sin(90° - A)$ [by IV.
$= -\cos(-A)$ [by III.
$= -\cos A$ [by II.

77. The method of Arts. 74, 75 may be easily applied to prove independently the result of Examples i. and ii. above, and it is **a much more instructive exercise** for the student to make out the independent proof for himself than to depend on a known result.

We give two examples which are important.

DEF. Two angles are said to be the **complement** the one of the other when their sum is 90°.

Thus $(90° - A)$ is the complement of A.

If $A+B+C=180°$ then $\left(\frac{A}{2}+\frac{B}{2}\right)$ is the complement of $\frac{C}{2}$.

Example i. *To prove that the sine of an angle = the cosine of its complement.*

That is, *to prove* $\sin A = \cos(90° - A)$

and $\cos A = \sin(90° - A)$.

We take **two revolving lines** OP and OP'. OP starting from OR is to describe the angle A; OP' starting from OR is to describe $(90° - A)$.

As usual, PM, $P'M'$ are perpendiculars on OR and $P'N'$ is a perpendicular on OU.

In describing $(90° - A)$ we shall consider that OP' starting from OR, turns first through 90° into the position OU, and then turns back from OU through the angle $UOP' = (-A)$.

So that ROP, the angle which OP describes from OR, is always equal to UOP', the angle which OP' describes from OU in the *opposite* direction.

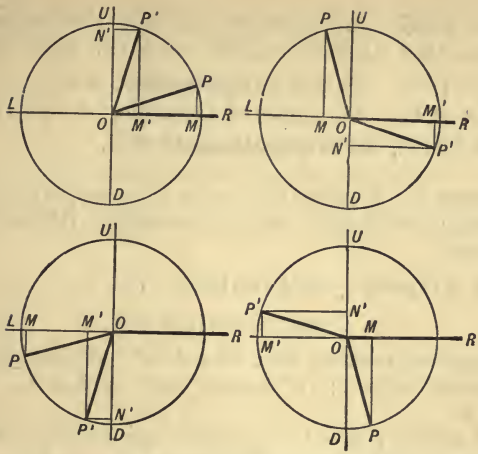
Hence, $N'P'$, that is OM', is *always* equal to MP in *magnitude*.

Also it will be seen that when P is *above* LOR, P' is to the *right* of UOD; when P is *below* LOR, P' is to the *left* of UOD.

Hence, OM' and MP have *always* the *same sign*.

Therefore $$\frac{MP}{OP} = \frac{OM'}{OP'} \text{ always,}$$

or, $\sin A = \cos(90° - A)$, for all values of A.



Again, ON', that is $M'P'$, is *always* equal to OM' in *magnitude.*

And P' is above or below LOR according as P is to the right or to the left of UOD.

So that $M'P'$ and OM have *always* the *same sign.*

Therefore $$\frac{OM}{OP} = \frac{M'P'}{OP'} \text{ always,}$$

or, $$\cos A = \sin (90° - A) \text{ for all values of } A.$$

EXAMPLES. XVII.

Prove, drawing a separate figure for each example, that

1. $\sin 30° = \cos 60°$.
2. $\sin 65° = \cos 25°$.
3. $\sin 195° = \cos (-105°)$.
4. $\cos 275 = \sin (-185°)$.
5. $\cos (-27°) = \sin 117°$.
6. $\cos 300° = \sin (-210°)$.

If A, B, C be the angles of a triangle, so that $A+B+C=180°$, prove

7. $\cos \frac{A}{2} = \sin \frac{B+C}{2}$.
8. $\cos \frac{B}{2} = \sin \frac{A+C}{2}$.
9. $\sin \frac{C}{2} = \cos \frac{A+B}{2}$.
10. $\sin \frac{A}{2} = \cos \frac{B+C}{2}$.

78. **Def.** Two angles are said to be the **Supplements** the one of the other when their sum is two right angles.

Thus $(180° - A)$ is the **supplement** of A.

If A, B, C be the angles of a triangle, $(A + B + C) = 180°$, so that $(B + C)$ is the **supplement** of A.

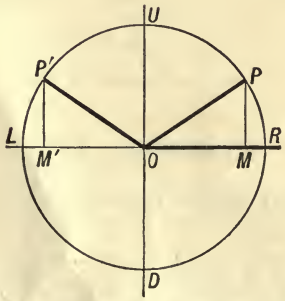
Example ii. *To prove that the sine of an angle = the sine of its supplement; and that the cosine of an angle = − (the cosine of its supplement).*

That is, *to prove* $\sin A = \sin (180° - A)$

and $\cos A = -\cos (180° - A)$.

We take **two revolving lines** OP and OP'. OP starting from OR describes the angle A; OP' starting from OR describes the angle $(180° - A)$.

In describing $(180° - A)$ we consider that OP' starting from OR turns first through 180° into the position OL, and then back from OL through the angle $LOP' = (-A)$.



So that ROP, the angle which OP describes from OR, is always equal to LOP', the angle which OP' describes from OL in the *opposite* direction.

Hence, MP and $M'P'$ are *always* equal in *magnitude*.

Also, P and P' are always *both* above, or *both* below LOR.
So that MP and $M'P'$ are *always* of the *same sign*.

Therefore $$\frac{MP}{OP} = \frac{M'P'}{OP'} \text{ always,}$$

or, $$\sin A = \sin (180° - A), \text{ for all values of } A.$$

Again, OM and OM' are *always* equal in *magnitude*.
Also it will be seen that when P is on the *right* of UOD, P' is on the *left* of UOD; when P is on the *left* of UOD, P' is on *right* of UOD.
So that OM and OM' are *always* of *opposite sign*.

Therefore $$\frac{OM}{OP} = -\frac{OM'}{OP'} \text{ always,}$$

or, $$\cos A = -\cos (180° - A), \text{ for all values of } A.$$

NOTE. By the method of Art. 76 we proceed thus

$$\sin (180° - A = \sin (90° + \overline{90° - A}) = \cos (90° - A)$$
$$= -\sin (-A) = \sin A$$
$$\cos (180° - A) = \cos (90° + \overline{90° - A}) = -\sin (90° - A)$$
$$= -\cos (-A) = -\cos A.$$

EXAMPLES. XVIII.

Prove, drawing a separate figure in each case, that

1. $\sin 60° = \sin 120°$.
2. $\sin 340° = \sin (-160°)$.
3. $\sin (-40°) = \sin 220°$.
4. $\cos 320° = -\cos (-140°)$.
5. $\cos (-380°) = -\cos 560°$.
6. $\cos 195° = -\cos (-15°)$.

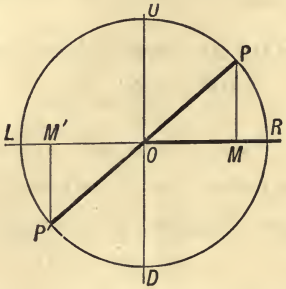
If A, B, C be the angles of a triangle, prove

1. $\sin A = \sin (B + C)$.
2. $\sin C = \sin (A + B)$.
3. $\cos B = -\cos (A + C)$.
4. $\cos A = -\cos (C + B)$.

Example iii. *To prove* $\sin A = -\sin(180° + A)$,
and $\cos A = -\cos(180° + A)$.

As before, we take **two revolving lines** OP and OP'. OP starting from OR describes the angle A; OP' starting from OR describes the angle $(180° + A)$.

In describing $(180° + A)$ we consider that OP' starting from OR turns first through 180° into the position OL, and then on from OL through the angle A.



So that ROP, the angle which OP describes from OR, is always equal to LOP', the angle which OP' describes from OL in the *same* direction.

Hence, $M'P'$ *always* $= MP$ in *magnitude*.

Also it will be seen that when P is *above* LOR, P' is *below* LOR and *vice versâ**.

So that MP and $M'P'$ are *always* of *opposite sign*.

Hence, $$\frac{MP}{OP} = -\frac{M'P'}{OP'} \text{ always,}$$

or, $\sin A = -\sin(180° + A)$, for all values of A.

Similarly, OM *always* $= OM'$ in *magnitude*.

And P' is to the left or to the right of UOD according as P is to the right or to the left of UOD.

Hence, $$\frac{OM}{OP} = -\frac{OM'}{OP'} \text{ always,}$$

or, $\cos A = -\cos(180° + A)$, for all values of A.

* This will be more clear if the student observes that POP' is always a straight line.

MISCELLANEOUS EXAMPLES. XIX.

Prove, drawing a separate figure in each case, that

1. $\sin 60° = -\sin 240°$.
2. $\sin 170° = -\sin 350°$.
3. $\sin(-20°) = -\sin 160°$.
4. $\cos 380° = -\cos 560°$.
5. $\cos(-225) = -\cos(-45°)$.
6. $\cos 1005° = -\cos 1185°$.
7. $\sin 60° = -\cos 150°$.
8. $\cos 60° = \sin 150°$.
9. $\sin 225 = -\cos 315°$.
10. $\cos(-60°) = \sin 30°$.

If $A+B+C$ be the angles of a triangle, prove that

11. $\sin A = -\sin(2A+B+C)$.
12. $\sin A = -\cos\dfrac{3A+B+C}{2}$.
13. $\cos B = \sin\dfrac{A+3B+C}{2}$.
14. $\cos C = -\cos(A+B+2C)$.
15. $\cos\dfrac{B-C}{2} = \sin\dfrac{A+2B}{2}$.
16. $\sin\dfrac{C-A}{2} = -\cos\dfrac{B+2C}{2}$.

Prove the following statements for all values of A and of B.

17. $\sin A = -\sin(-A)$.
18. $\cos A = \cos(-A)$.
19. $\sin A = \cos(A-90°)$.
20. $\cos A = -\sin(A-90°)$.
21. $\sin A = \cos(270°+A)$.
22. $\cos B = -\sin(270°+B)$.
23. $\sin C = -\cos(270°-C)$.
24. $\cos A = -\sin(270°-A)$.
25. $\sin(90°-B) = \sin(90°+B)$.
26. $\cos(180°+A) = \cos(180°-A)$.
27. $\tan(90°-A) = \cot A$.
28. $\tan A = -\tan(-A)$.
29. $\tan(90°+A) = -\cot(A)$.
30. $\tan A = -\tan(180°-A)$.
31. $\tan A = \tan(180°+A)$.
32. $\cot(90°-A) = \tan A$.

CHAPTER VII.

Solution of Trigonometrical Equations.

79. A *Trigonometrical* Equation is an equation in which there is a letter, such as θ, or x, which stands for an *angle* whose magnitude is sought.

The **solution** of such an equation is the process of finding angles, any one of which when it is substituted for θ, or x, satisfies the equation.

For example, $\cos\theta=\frac{1}{2}$ is a trigonometrical equation. Any angle whose cosine is $\frac{1}{2}$ satisfies the equation.

We know that $\cos 60°=\frac{1}{2}$.

Therefore $60°$ is a value for θ which satisfies the equation.

80. The **General Solution** of a Trigonometrical equation is the finding a formula which includes all angles each of which satisfies the equation.

Example i. A general solution of the equation $\cos\theta=\frac{1}{2}$ is

$$\theta=2n\times 180°\pm 60°$$

when n has any integral value positive or negative.

For by Art. 70 the cosine of each of the angles $2n\times 180°\pm 60°$ is $\frac{1}{2}$ and there is no angle whose cosine is $\frac{1}{2}$ which cannot be included in the expression $2n\times 180°\pm 60°$ by giving n the proper integral values.

Example ii. *Find a solution of the equation*

$$\sin\theta-\operatorname{cosec}\theta+\tfrac{3}{2}=0.$$

The usual method of solution is to express all the Trigonometrical Ratios in terms of one of them.

Thus we put $\dfrac{1}{\sin\theta}$ for $\operatorname{cosec}\theta$, and we get

$$\sin\theta-\frac{1}{\sin\theta}+\tfrac{3}{2}=0.$$

This is an equation in which θ, and therefore $\sin\theta$ is unknown. It will be convenient if we put x for $\sin\theta$, and then solve the equation for x as an ordinary algebraical equation. Thus we get

$$x-\frac{1}{x}+\tfrac{3}{2}=0,$$

or $$x^2+\frac{3x}{2}=1,$$

$$\therefore\ x^2+\frac{3x}{2}+\tfrac{9}{16}=1+\tfrac{9}{16}=\tfrac{25}{16}.$$

$$\therefore\ x+\tfrac{3}{4}=\pm\tfrac{5}{4}.$$

Whence $x=-2$, or $x=\frac{1}{2}$.

But x stands for $\sin\theta$.

Thus we get $\sin\theta=-2$, or $\sin\theta=\frac{1}{2}$.

The value -2 is inadmissible for $\sin\theta$, for there is no angle whose sine is numerically greater than 1 (Art. 54).

$$\therefore\ \sin\theta=\tfrac{1}{2}.$$

But $$\sin 30^\circ=\tfrac{1}{2}.$$

$$\therefore\ \sin\theta=\sin 30^\circ.$$

Therefore *one* angle which satisfies this equation for θ is 30°.

NOTE. A general solution of the above equation is a formula including every angle whose sine is $\frac{1}{2}$.

By Art. 68 $n\times 180^\circ+(-1)^n 30^\circ$, is a general solution.

Example iii. *Find a solution of the equation*

$$\operatorname{cosec}\theta-\cot^2\theta+1=0.$$

We have $$\cot^2\theta=\operatorname{cosec}^2\theta-1.$$

Therefore the given equation may be written

$$\operatorname{cosec}\theta-(\operatorname{cosec}^2\theta-1)+1=0,$$

or $$\operatorname{cosec}\theta-\operatorname{cosec}^2\theta+2=0.$$

Write x for $\operatorname{cosec}\theta$; then

$$x-x^2+2=0,$$

or $$x^2-x-2=0,$$

that is, $$x^2-x+\tfrac{1}{4}-\tfrac{1}{4}-2=0,$$

that is, $$(x-\tfrac{1}{2})^2-\tfrac{9}{4}=0,$$

that is, $$(x-\tfrac{1}{2}-\tfrac{3}{2})\,(x-\tfrac{1}{2}+\tfrac{3}{2})=0.$$

Whence either,

$$x-\tfrac{1}{2}-\tfrac{3}{2}=0\ ;\ \text{so that, } x=2 \quad\ldots\ldots\text{(i)},$$

or, $$x-\tfrac{1}{2}+\tfrac{3}{2}=0\ ;\ \text{so that, } x=-1 \quad\ldots\ldots\text{(ii)}.$$

But x stands for $\operatorname{cosec}\theta$; so that we have either

$$\operatorname{cosec}\theta=2 \ \text{ or } \ \operatorname{cosec}\theta=-1 \quad\ldots\ldots\text{(i)}.$$

But $\operatorname{cosec} 30^\circ=2$ and $\operatorname{cosec}(-90^\circ)=-1$ .....(ii).

Therefore 30° and -90° are each values of θ which satisfy the given equation.

Example iv. *Find the* general *solution of the above equation.*

The algebraic work is the same as before.

Every angle whose cosec is 2 satisfies the equation, and every angle whose cosec is -1 satisfies the equation.

Therefore the general solutions are every angle included in either of the expressions

$$n \times 180° + (-1)^n 30°, \quad \text{and} \quad (2n+1) \times 90°.$$

EXAMPLES. XX.

Find (*a*) one angle which satisfies each of the following equations.

(*b*) a general solution of each equation.

1. $\sin\theta = \frac{1}{\sqrt{2}}$.
2. $4\sin\theta = \operatorname{cosec}\theta$.
3. $2\cos\theta = \sec\theta$.
4. $4\sin\theta - 3\operatorname{cosec}\theta = 0$.
5. $4\cos\theta - 3\sec\theta = 0$.
6. $3\tan\theta = \cot\theta$.
7. $3\sin\theta - 2\cos^2\theta = 0$.
8. $\sqrt{2}\sin\theta = \tan\theta$.
9. $2\cos\theta = \sqrt{3}\cot\theta$.
10. $\tan\theta = 3\cot\theta$.
11. $\tan\theta + 3\cot\theta = 4$.
12. $\tan\theta + \cot\theta = 2$.
13. $2\sin^2\theta + \sqrt{2}\cos\theta = 2$.
14. $3\sin^2\theta + 2\sin\theta = 1$.
15. $3\tan^2\theta - 4\sin^2\theta = 1$.
16. $2\sin^2\theta + \sqrt{2}\sin\theta = 2$.
17. $\cos^2\theta - \sqrt{3}\cos\theta + \frac{3}{4} = 0$.
18. $\cos^2\theta + 2\sin^2\theta - \frac{5}{2}\sin\theta = 0$.

The Inverse Notation.

81. In the solution of trigonometrical equations we have to use such expressions as,

an angle whose sine is $\frac{1}{2}$.

It is therefore convenient to have an *abbreviation* for such an expression.

The notation employed is

$$\sin^{-1}\tfrac{1}{2}.$$

82. DEF. $\sin^{-1} a$ is an **angle** whose sine is a.

The solution of the equation $\sin\theta = a$ may be written

$$\theta = \sin^{-1} a.$$

Sometimes the meaning given to the expression $\sin^{-1} a$ is the *least positive angle whose sine is* a.

Example i. $30°$ is one value of $\sin^{-1}\frac{1}{2}$.
This is true because $\sin 30° = \frac{1}{2}$.

Example ii. $\sin(\sin^{-1}a) = a$.
Let $\theta = \sin^{-1}a$ then θ is an angle whose sine is a,
therefore $\sin\theta = a$, or $\sin(\sin^{-1}a) = a$.

*EXAMPLES. XXI.

Prove that the following statements are true when we take for $\sin^{-1}a$, etc. their least *positive* value.

1. $\sin^{-1}\frac{3}{5} = \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}$.

2. $\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{\sqrt{3}}{2} = \cot^{-1}\sqrt{3}$.

3. $\sin^{-1}a = \cos^{-1}\sqrt{1-a^2} = \tan^{-1}\frac{a}{\sqrt{1-a^2}}$.

83. The student must notice carefully that such a statement as $sin^{-1}\frac{1}{2} = cos^{-1}\frac{1}{2}\sqrt{3}$ is not an identity.

For $\sin^{-1}\frac{1}{2}$ is any one of the values of $n\pi + (-1)^n 30°$,
and $\cos^{-1}\frac{1}{2}\sqrt{3}$ is one of the values of $2n\pi \pm 30°$.

Thus $150°$ is one value of $\sin^{-1}\frac{1}{2}$, but $150°$ is not a value of $\cos^{-1}\frac{1}{2}\sqrt{3}$.

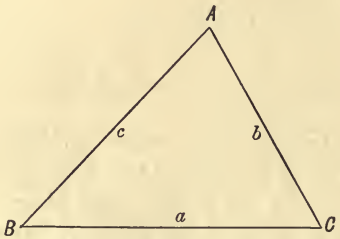
Example. Shew that the statements in Examples **1**, **2**, and **3** *of* Examples xxi. *are* not *true generally.*

CHAPTER VIII.

On the Relations between the Sides and Angles of a Triangle.

84. The **three sides** and the **three angles** of any **triangle** are called its six **parts.**

By the letters A, B, C we shall indicate *geometrically*, the three angular points of the triangle ABC; *algebraically*, the three angles at those angular points respectively.



By the letters a, b, c we shall indicate the measures of the sides BC, CA, AB opposite the angles A, B, C respectively.

85. **I.** We know that, $A + B + C = 180°$. [Euc. I. 32.]

86. Also if A be an angle of a triangle, then A may have any value between 0° and 180°. Hence,

(i) **sin A** must be positive (and less than 1),

(ii) **cos A** may be positive or negative (but must be numerically less than 1),

(iii) **tan A** may have any value whatever, positive or negative.

87. Also, if we are given the value of

(i) **sin A**, there are **two** angles, each less than 180°, which have the given positive value for their sine.

(ii) **cos A**, or (iii) **tan A**, then there is only **one** value of A, which value can be found from the Tables.

Example. *To prove* $\sin(A+B)=\sin C$.

$A+B+C=180°$, $\therefore\ A+B=180°-C$,

and $\therefore$ $\sin(A+B)=\sin(180°-C)=\sin C$. [p. 58.]

EXAMPLES. XXII.

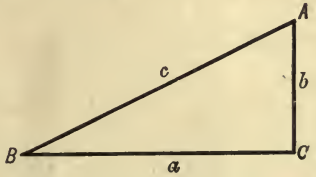
Find A from each of the six following equations, A being an angle of a triangle.

1. $\cos A=\frac{1}{2}$. 2. $\cos A=-\frac{1}{2}$. 3. $\sin A=\frac{1}{2}$.
4. $\tan A=-1$. 5. $\sin A=\frac{1}{2}\sqrt{2}$. 6. $\tan A=-\sqrt{3}$.

Prove the following statements, A, B, C being the angles of a triangle.

7. $\sin(A+B+C)=0$. 8. $\cos(A+B+C)=-1$.
9. $\sin\frac{A+B+C}{2}=1$. 10. $\cos\frac{A+B+C}{2}=0$.
11. $\tan(A+B)=-\tan C$. 12. $\cot\frac{B+C}{2}=\tan\frac{A}{2}$.
13. $\cos(A+B)=-\cos C$. 14. $\cos(A+B-C)=-\cos 2C$.

88. If ABC be a **right-angled** triangle having $C=90°$, then $(A+B)=90°$.



Hence, $\sin A=\sin(90°-B)=\cos B$. [p. 56.]

Also, $\sin A=\frac{a}{c}=\cos B$, and so on.

EXAMPLES. XXIII.

In a right-angled triangle ABC, in which C is a right angle, prove the following statements.

1. $\tan A=\cot B$. 2. $\tan B=\cot A+\cos C$.
3. $\sin 2A=\sin 2B$. 4. $\cos 2A+\cos 2B=0$.

89. II. *To prove* $a = b\cos C + c\cos B$.

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary.

There will be three cases. Fig. i. when both B and C are acute angles; Fig. ii. when one of them (B) is obtuse; Fig. iii. when one of them (B) is a right angle. Then

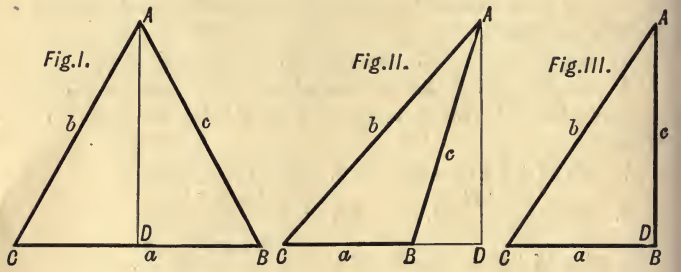


Fig. i. $\dfrac{CD}{CA} = \cos ACD$; or, $CD = b\cos C$,

and $\dfrac{DB}{AB} = \cos ABD$; or, $DB = c\cos B$,

$\therefore a = CD + DB = b\cos C + c\cos B$.

Fig. ii. $\dfrac{CD}{CA} = \cos ACD$; or, $CD = b\cos C$,

$\dfrac{BD}{AB} = \cos ABD$; or, $BD = c\cos(180° - B)$,

$\therefore a = CD - BD = b\cos C - c\cos(180° - B)$

$= b\cos C + c\cos B$.

Fig. iii. $a = CB = b\cos C$.

$= b\cos C + c\cos B$. [For, $\cos B = \cos 90° = 0$.]

Similarly it may be proved that,

$b = c\cos A + a\cos C$; $c = a\cos B + b\cos A$.

90. III. *To prove that in any triangle the sides are proportional to the sines of the opposite angles;* or, *To prove that* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary. Then,

Fig. i. $AD = b \sin C$; for, $\frac{AD}{AC} = \sin C$ [Def.]:

also $AD = C \sin B$; for, $\frac{AD}{AB} = \sin B$.

$\therefore b \sin C = c \sin B$;

or, $\frac{b}{\sin B} = \frac{c}{\sin C}$.

Fig. ii. $AD = b \sin C$,

and $AD = c \sin ABD = c \sin (180° - B)$

$\therefore AD = c \sin B$;

$\therefore b \sin C = c \sin B$;

or, $\frac{b}{\sin B} = \frac{c}{\sin C}$.

Fig. iii. $AB = AC \,.\, \sin C$; or, $c = b \sin C$;

$\therefore \frac{c}{\sin C} = \frac{b}{\sin B}$. [For $\sin B = \sin 90° = 1$.]

Similarly it may be proved that

$$\frac{a}{\sin A} = \frac{b}{\sin B};$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$ Q.E.D.

91. IV. *To prove that* $a^2 = b^2 + c^2 - 2bc \cos A$.

Take one of the angles A. Then of the other two, one must be acute. Let B be an acute angle. From C draw CF perpendicular to BA, or to BA produced if necessary.

There will be three figures according as A is less, greater than, or equal to a right angle. Then,

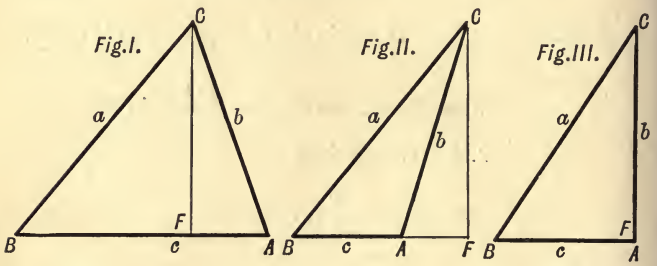


Fig. i. $BC^2 = CA^2 + AB^2 - 2 \,.\, BA \,.\, FA$; [Euc. II. 13]

or, $a^2 = b^2 + c^2 - 2c \,.\, FA$

$= b^2 + c^2 - 2cb \cos A$. [For $FA = b \,.\, \cos A$.]

Fig. ii. $BC^2 = CA^2 + AB^2 + 2 \,.\, BA \,.\, AF$; [Euc. II. 12]

or, $a^2 = b^2 + c^2 + 2cb \cos FAC$

$= b^2 + c^2 - 2bc \cos A$. [For $FAC = 180° - A$.]

Fig. iii. $BC^2 = CA^2 + AB^2$; [Euc. I. 47]

or, $a^2 = c^2 + c^2 - 2bc \cos A$. [For $\cos A = \cos 90° = 0$.]

Similarly it may be proved that

$$b^2 = c^2 + a^2 - 2ca \cos B,$$

and that $$c^2 = a^2 + b^2 - 2ab \cos C.$$

92. V. Hence,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

Example. Suppose we are given that

$$\left.\begin{aligned} a &= b\cos C + c\cos B, \\ b &= c\cos A + a\cos C, \\ c &= a\cos B + b\cos A. \end{aligned}\right\}$$

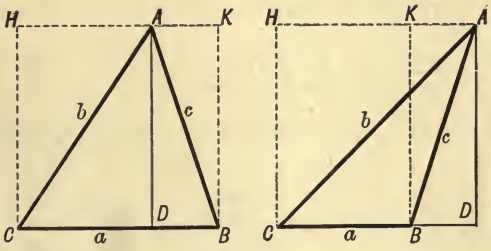
Then, taking a times the first $+b$ times the second $-c$ times the third, we get

$$a^2+b^2-c^2=(ab\cos C+ac\cos B)+(bc\cos A+ba\cos C) \\ -(ca\cos B+cb\cos A),$$

$$=2ab\cos C.$$

93. *To find an expression for the* **Area** *of a Triangle.*

The area of the triangle ABC is denoted by Δ.



Through A draw HK parallel to BC, and through ABC draw lines AD, BK, CH perpendicular to BC.

The area of the triangle ABC is half that of the rectangular parallelogram $BCHK$. [Euc. I. 41.]

Therefore $$\Delta=\frac{BC\,.\,CH}{2}=\frac{BC\,.\,DA}{2}$$

$$=\frac{a\,.\,b\sin C}{2}=\tfrac{1}{2}abc\,\frac{\sin C}{c}.$$

94. The student is advised to make himself thoroughly familiar with the following **formulæ :**

$$A+B+C=180^\circ \quad \text{.....(i)},$$

$$a=b\cos C+c\cos B \quad \text{.....(ii)},$$

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{abc}{2\Delta} \quad \text{.....(iii)},$$

$$a^2=b^2+c^2-2bc\cos A \quad \text{.....(iv)},$$

$$\cos A=\frac{b^2+c^2-a^2}{2bc} \quad \text{.....(v)}.$$

EXAMPLES. XXIV.

In any triangle ABC prove the following statements :

1. $\dfrac{\sin A+2\sin B}{a+2b}=\dfrac{\sin C}{c}$.

2. $\dfrac{\sin^2 A-m.\sin^2 B}{a^2-m.b^2}=\dfrac{\sin^2 C}{c^2}$.

3. $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$.

4. $b+c-a=(b+c)\cos A-(c-a)\cos B+(a-b)\cos C$.

5. $a\cos(A+B+C)-b\cos(B+A)-c\cos(A+C)=0$.

6. $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}$.

CHAPTER IX.

On the Solution of Triangles.

95. The problem known as the **Solution of Triangles** may be stated thus: *When a sufficient number of the parts of a triangle are given, to find the magnitude of each of the other parts.*

96. When **three** parts of a Triangle (one of which must be a side) are given, the other parts can in general be determined.

There are four cases.

I. Given **three sides.** [Compare Euc. I. 8.]

II. Given **one side** and **two angles.** [Euc. I. 26.]

III. Given **two sides** and the **angle between** them. [Euc. I. 4.]

IV. Given **two sides** and the **angle opposite** one of them. [Compare Euc. VI. 7.]

97. Practical measurements are always approximate and are expressed in 3, 4, or 5 figures according to the accuracy of the measurement. Hence calculations are almost invariably made by the aid of logarithms.

The solutions given below in Cases I. and III. are not the solutions usually used. We give them, however, as the methods of solution adapted to logarithmic computation require a knowledge of compound angles.

Case I.

98. Given **three sides**, a, b, c. [Euc. I. 8; VI. 5.]

This Case may be solved by the formula

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

But this formula is not adapted for logarithmic calculation, and therefore is seldom used in practice.

It may sometimes be used with advantage, when the given lengths of a, b, c each contain less than three digits.

Example. Find the greatest angle of the triangle whose sides are 13, 14, 15.

Let $a=15$, $b=14$, $c=13$. Then the greatest angle is A.

Now, $\cos A = \dfrac{14^2+13^2-15^2}{2\times 14\times 13} = \dfrac{140}{2\times 14\times 13} = \frac{5}{13} = \cdot 384615.$

$= \cos 67° 23'$, nearly.

[By the Table of natural cosines.]

$\therefore$ the greatest angle $= 67° 23'$.

EXAMPLES. XXV.

1. The sides of a triangle are 2, $\sqrt{6}$, and $1+\sqrt{3}$, find the angles.

2. The sides of a triangle are 2, $\sqrt{2}$ and $\sqrt{3}-1$, find the angles.

Case II.

99. Given **one side** and **two angles**, as a, B, C.

[Euc. I. 26; VI. 4.]

First, $A = 180° - B - C$; which determines A.

Next, $$\frac{b}{\sin B} = \frac{a}{\sin A},$$

or, $$b = \frac{a \,.\, \sin B}{\sin A};$$

and, $$\frac{c}{\sin C} = \frac{a}{\sin A},$$

or, $$c = \frac{a \,.\, \sin C}{\sin A}.$$

These determine b and c.

Case III.

100. The formula $a^2 = b^2 + c^2 - 2bc \cos A$ may be used in simple cases.

Example. If $b = 35$ feet, $c = 21$ feet, and $A = 50°$, find a, given that $\cos 50° = \cdot 643$.

Here $$a^2 = 35^2 + 21^2 - 2 \times 35 \times 21 \times \cos 50°;$$

$$\therefore \frac{a^2}{7^2} = 5^2 + 3^2 - 2 \times 5 \times 3 \times \cos 50°,$$

$$= 25 + 9 - 30 \times \cdot 643, \quad = 14 \cdot 71.$$

$$\therefore \frac{a}{7} = 3 \cdot 82 \text{ nearly; or, } a = 26 \cdot 74 = \text{about } 26\tfrac{3}{4} \text{ feet.}$$

EXAMPLE. XXVI.

1. If $a = 21$ chains, $b = 20$ chains, $C = 60°$, find c.

Case IV.

101. Given **two sides** and the **angle opposite** one of them, as b, c, B. [Omitted in Euc. I.; Euc. VI. 7.]

First, since $\frac{c}{\sin C} = \frac{b}{\sin B}$; $\therefore \sin C = \frac{c \sin B}{b}$.

C must be found from this equation.

When C is known, $A = 180° - B - C$,

and, $a = \frac{b \sin A}{\sin B}$, which solves the triangle.

102. We remark however that the angle C, found from the **trigonometrical** equation *sin* C = *a given quantity*, where C is an angle of a triangle, has **two** values, one less than 90°, and one greater than 90°. [Art. 86.]

The question arises, Are both these values admissible?

This may be decided as follows:

If B is not less than 90°, C must be less than 90°; and the smaller value for C **only** is admissible.

If B is less than 90° we proceed thus.

1. If b is less than $c \sin B$, then $\sin C$, which $= \frac{c \sin B}{b}$, is greater than 1. This is impossible. Therefore if b is less than $c \sin B$, there is **no** solution whatever.

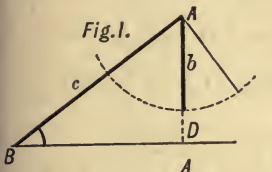
2. If b is equal to $c \sin B$, then $\sin C = 1$, and therefore $C = 90°$; and there is only **one** value of C, viz. 90°.

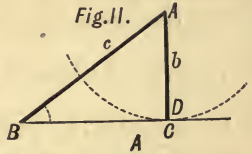
3. If b is greater than $c \sin B$, and less than c, then B is less than C, and C may be obtuse or acute. In this case C may have either of the values found from the equation $\sin C = \frac{c \sin B}{b}$. Hence there are **two** solutions, and the triangle is said to be **ambiguous.**

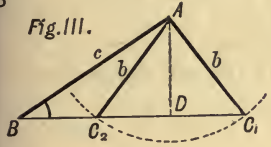
4. If b is equal to or greater than c, then B is equal to or greater than C, so that C must be an acute angle; and the smaller value for C **only** is admissible.

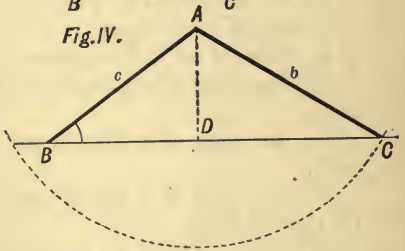
103. The same results may be obtained **geometrically.**

Construction. Draw $AB = c$; make the angle ABD = the given angle B; with centre A and radius $= b$ describe a circle; draw AD perpendicular to BD.









Then $AD = c \sin B$.

1. If b is less than $c \sin B$, i.e. less than AD, the circle will not cut BD at all, and the construction **fails.** (Fig. i.)

2. If b is equal to AD, the circle will *touch* the line BD in the point D, and the required triangle is the **right-angled** triangle ABD. (Fig. ii.)

3. If b is greater than AD and less than AB, i.e. than c, the circle will cut the line BD in *two* points C_1, C_2 each on the same side of B. And we get **two** triangles ABC_1, ABC_2 each satisfying the given condition. (Fig. iii.)

4. If b is equal to c, the circle cuts BD in B and in one other point C; if b is greater than c the circle cuts BD in *two* points, but on *opposite* sides of B. In either case there is only **one** triangle satisfying the given condition. (Fig. iv.)

MISCELLANEOUS EXAMPLES. XXVII.

1. Simplify the formulæ $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, in the case of an equilateral triangle.

2. The sides of a triangle are as $2 : \sqrt{6} : 1+\sqrt{3}$, find the angles.

3. The sides of a triangle are as $4, 2\sqrt{2}, 2(\sqrt{3}-1)$, find the angles.

4. Given $C=120°$, $c=\sqrt{19}$, $a=2$, find b.

5. Given $A=60°$, $b=4\sqrt{7}$, $c=6\sqrt{7}$, find a.

6. Given $A=45°$, $B=60°$ and $a=2$, find c.

7. The sides of a triangle are as $7 : 8 : 13$, find the greatest angle.

8. The sides of a triangle are $1, 2, \sqrt{7}$, find the greatest angle.

9. The sides of a triangle are as $a : b : \sqrt{(a^2+ab+b^2)}$, find the greatest angle.

10. When $a : b : c$ as $3 : 4 : 5$, find the greatest and least angles; given $\cos 36° 52' = \cdot 8$.

11. If $a=5$ miles, $b=6$ miles, $c=10$ miles, find the greatest angle. [$\cos 49° 33' = \cdot 65$.]

12. If $a=4$, $b=5$, $c=8$, find C; given that $\cos 54° 54' = \cdot 575$.

13. The two sides a and b are $\sqrt{3}+1$ and 2 respectively; $C=30°$; solve the triangle.

14. Given $C=18°$, $a=\sqrt{5}+1$, $c=\sqrt{5}-1$, find the other angles.

15. If $b=3$, $C=120°$, $c=\sqrt{13}$, find a and the sines of the other angles.

16. Given $A=105°$, $B=45°$, $c=\sqrt{2}$, solve the triangle.

17. Given $B=75°$, $C=30°$, $c=\sqrt{8}$, solve the triangle.

18. Given $B=45°$, $c=\sqrt{75}$, $b=\sqrt{50}$, solve the triangle.

19. Given $B=30°$, $c=150$, $b=50\sqrt{3}$, shew that of the two triangles which satisfy the data, one will be isosceles and the other right-angled. Find the third side in the greatest of these triangles.

20. Is the solution ambiguous when $B=30°$, $c=150$, $b=75$?

21. If the angles adjacent to the base of a triangle are $22\frac{1}{2}°$ and $112\frac{1}{2}°$, shew that the perpendicular altitude will be half the base.

22. If $a=2$, $b=4-2\sqrt{3}$, $c=\sqrt{6}(\sqrt{3}-1)$, solve the triangle.

23. If $A=9°$, $B=45°$, $b=\sqrt{6}$, find c.

24. Given $B=15°$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$, solve the triangle.

25. Given $\sin B=\cdot25$, $a=5$, $b=2\cdot5$, find A. Draw a figure to explain the result.

26. Given $C=15°$, $c=4$, $a=4+\sqrt{48}$, solve the triangle.

27. Two sides of a triangle are $3\sqrt{6}$ yards and $3(\sqrt{3}+1)$ yards, and the included angle $45°$, solve the triangle.

28. If $C=30°$, $b=100$, $c=45$, is the triangle ambiguous?

29. Prove that if $A=45°$ and $B=60°$ then $2c=a(1+\sqrt{3})$.

30. The cosines of two of the angles of a triangle are $\frac{1}{2}$ and $\frac{3}{5}$, find the ratio of the sides.

CHAPTER X.

On the Measurement of Heights and Distances.

104. We have said (Art. 36) that the measurement, with scientific accuracy, of a line of any considerable length involves a long and difficult process.

On the other hand, sometimes it is required to find the *direction* of a line that it may point to an object which is

not visible from the point from which the line is drawn. As for example when a tunnel has to be constructed.

By the aid of the Solution of Triangles

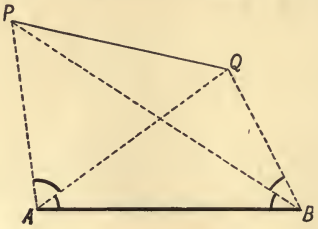
we can find the length of the distance between points which are inaccessible ;

we can calculate the magnitude of angles which cannot be practically observed ;

we can find the relative heights of distant and inaccessible points.

The method on which the Trigonometrical Survey of a country is conducted affords the following illustration.

105. *To find the distance between two distant objects.*



Two convenient positions A and B, on a level plain as far apart as possible, having being selected, the distance between A and B is measured with the greatest possible care. This line AB is called the **base** line. (In the survey of England, the base line is on Salisbury plain, and is about 36,578 feet long.)

Next, the two distant objects, P and Q (church spires, for instance) visible from A and B, are chosen.

The angles PAB, PBA are observed. Then by Case II. Chapter IX., the lengths of the lines PA, PB are calculated.

Again, the angles QAB, QBA are observed; and by Case II. the lengths of QA and QB are calculated.

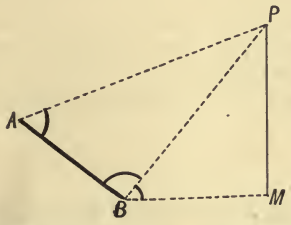
Thus the lengths of PA and QA are found.

The angle PAQ is observed; and then by Case III. the length of PQ is calculated.

106. Thus the distance between two points P and Q has been found. The points P and Q are not necessarily accessible; the only condition being that P and Q must be *visible* from both A and B.

107. In practice, the points P and Q will generally be accessible, and then the line PQ, whose length has been calculated, may be used as a new base to find other distances.

108. *To find the height of a distant object above the point of observation.*



Let B be the point of observation; P the distant object. From B measure a base line BA of any convenient length, in any convenient direction; observe the angles PAB, PBA, and by Case II. calculate the length of BP. Next observe at B the 'angle of elevation' of P; that is, the angle which the line BP makes with the horizontal line BM, M being the point in which the vertical line through P cuts the horizontal plane through B.

Then PM, which is the vertical height of P above B can be calculated, for $PM = BP \,.\, \sin MBP$.

EXAMPLES. XXVIII.

1. Two straight roads inclined to one another at an angle of 60°, lead from a town A to two villages B and C; B on one road distant 30 miles from A, and C on the other road distant 15 miles from A. Find the distance from B to C. Ans. 25·98 m.

2. Two ships leave harbour together, one sailing N.E. at the rate of $7\frac{1}{2}$ miles an hour and the other sailing North at the rate of 10 miles an hour. Prove that the distance between the ships after an hour and a half is 10·6 miles.

3. A and B are two consecutive milestones on a straight road and C is a distant spire. The angles ABC and BAC are observed to be 120° and 45° respectively. Shew that the distance of the spire from A is 3·346 miles.

4. If the spire C in the last question stands on a hill, and its angle of elevation at A is 15°, shew that it is ·866 of a mile higher than A.

5. If in Question 3 there is another spire D such that the angles DBA and DAB are 45° and 90° respectively and the angle DAC is 45°; prove that the distance from C to D is $2\frac{3}{4}$ miles very nearly.

QUESTIONS SET IN THE EXAMINATION FOR THE ADDITIONAL SUBJECTS OF THE PREVIOUS EXAMINATION AT CAMBRIDGE.

I A. *October*, 1886.

1. Explain the measurement of angles in degrees, minutes and seconds.

How many minutes of angle does the hour hand of a watch pass over in a minute of time?

2. Shew from the definitions of the Trigonometrical Functions that $\sin^2 A + \cot^2 A + \cos^2 A = \operatorname{cosec}^2 A$.

Prove also that $\dfrac{\tan A + \sec A + 1}{\tan A + \sec A - 1} = \dfrac{\sec A + 1}{\tan A}$.

3. Find the values of the sine, cosine and tangent of 30°.

In a triangle ABC, the angle C is a right angle, the angle A is 60°, and the length of the perpendicular let fall from C on AB is 20 feet; find the length of AB.

4. Trace the changes in $\sin\theta$ as θ changes from 0° to 360°.

If $\sin\theta = \frac{1}{2}\sqrt{3}$ write down the four possible values of $\sin\frac{1}{2}\theta$.

I B. *October*, 1886.

1. Explain the measurement of angles in degrees, minutes and seconds.

How many seconds of angle does the hour hand of a watch pass over in a second of time?

2. Shew from the definitions of the Trigonometrical Functions that $\sin^2 A + \tan^2 A + \cos^2 A = \sec^2 A$.

Prove also that $\dfrac{\cot A + \operatorname{cosec} A + 1}{\cot A + \operatorname{cosec} A - 1} = \dfrac{\operatorname{cosec} A + 1}{\cot A}$.

3. Find the values of the sine, cosine and tangent of 60°.

In a triangle ABC, the angle A is a right angle, the angle C is 30°, and the length of the perpendicular let fall from A on BC is 12 feet; find the length of BC.

4. Trace the changes in $\cos\theta$ as θ changes from 0° to 360°.

If $\sin\theta = \frac{1}{2}\sqrt{3}$, write down the four possible values of $\cos\frac{1}{2}\theta$.

II A. *December*, 1886.

1. Define the secant and cotangent of an angle and prove

(i) that $\sec\theta \operatorname{cosec}\theta = \tan\theta + \cot\theta$.

(ii) that $\cot^2\theta \dfrac{\sec\theta - 1}{1 + \sin\theta} + \sec^2\theta \dfrac{\sin\theta - 1}{1 + \sec\theta} = 0$.

2. From the definitions of the Trigonometrical Functions prove that $\sin^2 A = 1 - \cos^2 A$, and $\sin A \tan A = \sec A - \cos A$.

Prove that $\frac{1}{3}(\cos^6 A + \sin^6 A) - \frac{1}{4}(\cos^2 A - \sin^2 A)^2 = \frac{1}{12}$.

3. Investigate the values of $\tan 45°$ and $\sin 60°$.

Two adjacent sides of a parallelogram are of lengths 15 and 24 and the angle between them is 60°; find the lengths of both diagonals.

II B. *December*, 1886.

1. Define the tangent and cosecant of an angle and prove

(i) that $\sec^2\theta + \operatorname{cosec}^2\theta = (\tan\theta + \cot\theta)^2$.

(ii) that $\tan^2\theta \dfrac{\operatorname{cosec}\theta - 1}{1 + \cos\theta} + \operatorname{cosec}^2\theta \dfrac{\cos\theta - 1}{1 + \operatorname{cosec}\theta} = 0$.

2. From the definitions of the Trigonometrical Functions prove that $\cos^2 A + \sin^2 A = 1$, and $\operatorname{cosec}^2 A \tan^2 A = 1 + \tan^2 A$.

Prove that $4(\cos^6 A + \sin^6 A) - 3(\cos^4 A - \sin^4 A)^2 = 1$.

3. Investigate the values of cot 45° and cos 30°.

Two adjacent sides of a parallelogram are of lengths 18 and 25 and the angle between them is 120°; find the lengths of both diagonals.

III A. *June*, 1887.

1. Explain the mode of measuring angles in degrees, minutes and seconds.

Find the number of seconds of angle through which the earth revolves about its axis in a second of time.

2. Define the sine and tangent of an angle; shew how to find the sine and tangent of an angle whose cosine (m) is given.

If $\sin A = \tan B$ prove that

$$\cos^2 A \cos^2 B = (\cos B + \sin B)(\cos B - \sin B).$$

3. Find the values of sec 60° and of sec 45°.

Solve the equation $2 \operatorname{cosec} A = 2 \sin A + \cot A$.

4. Trace the changes in the tangent of an angle as the angle changes from 180° to 270°.

If $\sin\theta = -\frac{2}{3}$ find $\tan\theta$; explain by means of a figure, the reason why there are two answers to this question.

III B. *June*, 1887.

1. Explain the mode of measuring angles in degrees, minutes and seconds.

Find the number of minutes of angle through which the earth revolves about its axis in a minute of time.

2. Define the cosine and cotangent of an angle; shew how to find the cosine and tangent of an angle whose sine (m) is given.

3. Find the values of cosec 60° and cosec 45°.

Solve the equation $2 \sec A = 2 \cos A + \tan A$.

4. Trace the changes in the tangent of an angle as the angle changes from 270° to 360°.

If $\sin\theta = -\frac{3}{4}$, find $\tan\theta$; and explain by means of a figure the reason why there are two answers to the question.

IV A. *October*, 1887.

1. One angle of a triangle contains 80° 12′ 45″; one of the other angles is equal to twice the third angle; find the other two angles.

2. Define the sine and tangent of an angle and express each of them in terms of the secant.

Prove that

$$\sec^2 A \operatorname{cosec}^2 A - \tan^2 A)(\sec^2 A \operatorname{cosec}^2 A - \cot^2 A) = 1 + 2 \sec^2 A \operatorname{cosec}^2 A.$$

3. Find the values of sin 60°, cos 0°, and tan 90°.

Find a value of x if $\sin x^\circ + \operatorname{cosec} x^\circ = \frac{5}{2}$.

4. Trace the changes in sign and magnitude of the cosine of an angle as the angle increases from 0° to 180°.

If the sine of an angle be $> \frac{1}{2}$ and the cosine of that angle be $> \frac{1}{\sqrt{2}}$, between what limits does the angle lie?

IV B. *October*, 1887.

1. One angle of a triangle contains 46° 59′ 12″ and one of the other angles is equal to three times the third angle; find the other two angles.

2. Define the cosine and cotangent of an angle and express each of them in terms of the cosecant.

Prove that $(\sec^2 A + \tan^2 A)(\operatorname{cosec}^2 A + \cot^2 A) = 1 + 2\sec^2 A \operatorname{cosec}^2 A$.

3. Find the values of $\sin 30°$, $\sin 90°$, and $\cot 0°$.

Find a value of x when $\cos x° + \sec x° = \frac{5}{2}$.

4. Trace the changes in sign and magnitude of the sine of an angle as the angle increases from 0° to 180°.

If the sine of an angle be $> \frac{1}{\sqrt{2}}$ and the cosine of an angle be $> \frac{1}{2}$, between what limits does the angle lie?

V A. *December*, 1887.

1. Define the cosine and cosecant of an angle, and find the greatest and least values of each of those ratios.

Express each of those ratios in terms of the other.

2. Find the values of $\sin 45°$, $\sin 60°$, and $\sin 90°$.

If $\sin(4A + 3B) = 2\sin(3A + B) = 1$, find A and B.

3. What is meant by the complement of an angle?

If the trigonometrical ratios of an angle be known find those of the complement.

4. Trace the changes in the sign and magnitude in the tangent of an angle as the angle increases from 0° to 180°.

If $3\sin A + 2\cos A = 3$, find the value of $\tan A$.

V B. *December*, 1887.

1. Define the sine and secant of an angle, and find the greatest and least values of each of those ratios.

Express each of those ratios in terms of the other.

2. Find the values of $\cos 0°$, $\cos 30°$, $\cos 45°$.

If $\cos(4A - 3B) = 2\cos(3A - B) = 1$, find A and B.

3. What is meant by the supplement of an angle?

If the trigonometrical ratios of an angle be known find those of its supplement.

4. Trace the changes in the sign and magnitude of the cotangent of an angle as the angle increases from 0° to 180°.

If $3\cos A + 2\sin A = 3$, find the value of $\cot A$.

VI A. *June*, 1888.

1. Define the tangent and cosecant of an angle, and express each of them in terms of the cosine.

Prove that $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

2. Arrange in order of magnitude $\sin 17°$, $\cos 17°$, $\sec 17°$ and $\operatorname{cosec} 17°$; proving that the order you give is correct.

Shew that there is an angle less than a right angle whose cosine is equal to its tangent. Find the value of the sine of this angle correct to 3 places of decimals.

3. Prove that $\sin(180° - A) = \sin A$ and $\cos(180° - A) = -\cos A$.

4. Trace the change in the magnitude and sign of the tangent of an angle as the angle varies from 0° to 180°.

Find all the angles between 0° and 1000° which satisfy the equation $\tan\theta = \frac{1}{\sqrt{3}}$.

5. If $\frac{\sin\alpha}{\sin\beta} = \sqrt{2}$ and $\frac{\tan\alpha}{\tan\beta} = \sqrt{3}$, find α and β.

VI B. *June*, 1888.

1. Define the cotangent and secant of an angle, and express each of them in terms of the sine.

Prove that $\cot^2 A \,.\, \cos^2 A = \cot^2 A - \cos^2 A$.

2. Arrange in order of magnitude $\sin 57°$, $\cos 57°$, $\sec 57°$ and $\operatorname{cosec} 57°$, proving that the order you give is correct.

Shew that there is an angle less than a right angle whose sine is equal to its cotangent. Find the value of the cosine of this angle correct to 3 places of decimals.

3. Prove that $\sin(180° + A) = -\sin A$ and $\cos(180° + A) = -\cos A$.

4. Trace the changes in the magnitude and sine of the cotangent of an angle as the angle varies from 0° to 360°.

Find all the angles between 0° and 1000° which satisfy the equation $\cot\theta = \sqrt{3}$.

5. If $\frac{\cos\alpha}{\cos\beta} = \sqrt{\frac{2}{3}}$ and $\frac{\cot\alpha}{\cot\beta} = \frac{1}{\sqrt{3}}$, find α and β.

VII A. *October*, 1888.

1. Define the tangent, secant and cosecant of an angle; and from your definitions prove that

$$\sec A \operatorname{cosec} A = \tan A + \cot A.$$

Express $\sec A$ in terms of $\operatorname{cosec} A$.

2. Find the values of $\cos 45°$, $\tan 60°$ and $\operatorname{cosec} 0°$.

Write down all the values of A between 500 and 1000 for which $\sin A° = -\frac{1}{2}$.

3. Define the terms complement and supplement of an angle. If α be the complement of the supplement of any angle and β the supplement of the complement of the same angle, prove that

$$\sin\alpha = -\sin\beta \text{ and } \tan\alpha = \tan\beta.$$

4. If the trigonometrical ratios of an angle of $A°$ be known, shew how to find those of the angle $(180° + A°)$.

VII B. *October*, 1888.

1. Define the tangent, secant, cotangent and cosecant of an angle and from your definitions prove that

$$\tan\theta \operatorname{cosec}^2\theta = \tan\theta + \cot\theta.$$

2. Find the values of $\sin 45°$, $\tan 30°$ and $\operatorname{cosec} 90°$.

Write down all the values of A between 1000 and 1500 for which $\sin A° = -\frac{1}{2}$.

3. Define the terms complement and supplement of an angle.

If α be the complement of the supplement of an angle and β the supplement of the complement of the same angle prove that

$$\cos\alpha = -\cos\beta \text{ and } \cot\alpha = \cot\beta.$$

4. If the trigonometrical ratios of an angle of $A°$ be known, shew how to find those of the angles $(90° + A°)$.

VIII A. *December*, 1888.

1. There are two equilateral and equiangular polygons one of which has twice as many sides as the other; each angle of the former is one half as large again as an angle of the latter. Find the number of degrees in an angle of each polygon.

2. Define the tangent, cotangent, secant and cosecant of an angle, and prove that $\sec^2 A \operatorname{cosec}^2 A - \tan^2 A - \cot^2 A = 2$.

3. Trace the changes in sign and value of $\sec A$ as A changes from 0° to 360°.

4. Find the values of the cosine and cotangent of 120° and write down all the values from 0° to 1000° which satisfy the equation $\sin^2\theta = \frac{3}{4}$.

5. Find in each case the least positive value of A which will satisfy the equations

(i) $5\tan A° + 6\cot A° = 11$.

(ii) $2\sin(90° + A°)\sin(90° - A°) - 3\cos(90° + A°)\cos(90° - A°) = 2\frac{3}{4}$.

VIII B. *December*, 1888.

1. There are two equilateral and equiangular polygons one of which has twice as many angles as the other; each angle of the former is one third as large again as an angle of the other. Find the number of degrees in an angle of each polygon.

2. Define the sine, cosine, tangent and cotangent of an angle and prove that $\tan A - \cot A = (\tan A - 1)(\cot A + 1)$.

3. Trace the changes in sign and value of cosec $A°$ as A changes from 0° to 360°.

4. Find the values of the sine and tangent of 150°, and write down all values of θ between 0° and 1000° which satisfy the equation $\cos^2\theta = \frac{3}{4}$.

5. Find in each case the least positive value of A which will satisfy the equations

(i) $3\tan A° + 4\cot A° = 7$.

(ii) $3\sin(90°+A°)\sin(90°-A°) + 2\cos(90°+A°)(\cos 90°-A°) = 1\frac{3}{4}$.

IX A. *June*, 1889.

1. Explain how angles are measured in degrees, minutes and seconds.

How often is the angle 1° 4′ 22″ contained in the angle 28° 57′ 54″?

2. Define the sine and tangent of an angle, and explain each in terms of the secant.

Prove the relation

$$(\tan A + \cot A + \sec A)(\tan A + \cot A - \sec A) - 1 = \cot^2 A.$$

3. What is the measure (in degrees etc.) of the angle contained by the hands of a watch at a quarter past two o'clock?

4. Shew how to construct an angle whose cosine is $\frac{7}{8}$. Is the angle so constructed greater or less than 30°?

IX B. *June*, 1889.

1. How often does the angle 59° 44′ 42″ contain the angle 2° 50′ 42″?

2. Define the cosine and cotangent of an angle, and express each in terms of the cosecant.

Prove the relation

$$(\tan A + \cot A + \operatorname{cosec} A)(\tan A + \cot A - \operatorname{cosec} A) - 1 = \tan^2 A.$$

3. What is the measure (in degrees etc.) of the angle contained by the hands of a watch at a quarter to ten o'clock?

4. Shew how to construct an angle whose cotangent is $\frac{7}{4}$. Is the angle so constructed greater or less than 30°?

X A. *October*, 1889.

1. Define the sine and cosine of an angle.

Find values of θ which satisfy the equation

$$4\cos^2\theta - 8\sin\theta + 1 = 0.$$

2. Trace the changes in the sign and magnitude of the sine of an angle as the angle increases from 0° to 360°.

Shew that the acute angle whose sine is $\frac{1}{\sqrt{3}}$ lies between 30° and 45°.

3. Express the tangent of an angle in terms (i) of its cosine and (ii) of its cosecant.

If $\tan\theta = 2$, find $\cos\theta$ and $\operatorname{cosec}\theta$.

4. If the trigonometrical ratios be known for the angle $A°$, find those for the angle $(180°+A°)$.

Write down all the values of A between 500 and 1000 for which $\sin A° = -\frac{1}{2}\sqrt{3}$, giving in each case the corresponding value of the cosine.

X B. *October*, 1889.

1. Define the sine and cosine of an angle.

Find values of θ which satisfy the equation

$$4\sin^2\theta - 8\cos\theta + 1 = 0.$$

2. Trace the changes in the sign and magnitude of the cosine of an angle as the angle increases from 0° to 360°.

Shew that the acute angle whose cosine is $\frac{1}{\sqrt{3}}$ lies between 45° and 60°.

3. Express the tangent of an angle in terms (i) of its sine, (ii) of its secant.

If $\tan\theta = 3$, find $\sin\theta$ and $\sec\theta$.

4. If the trigonometrical ratios are known for the angle $A°$, find those for the angle $(90°+A°)$.

Write down all the values of A between 500 and 1000 for which $\cos A° = -\frac{1}{2}\sqrt{3}$, giving in each case the corresponding value of the sine.

XI A. *December*, 1889.

1. Shew that the times required by the minute and hour hands of a watch to turn through an angle of one degree are 10 seconds and 2 minutes respectively.

Find the angle between the hands of a watch at four minutes to two o'clock.

2. Explain how angles are measured.

Find the measures of the three angles of an isosceles triangle in which each of the two equal angles is one fourth of the third angle.

3. Find the values of $\sec 30°$ and $\cot 45°$.

Find values of $\tan\theta$ and $\sec\theta$ which satisfy the equation

$$3(\sec\theta - \tan\theta) = 2.$$

4. Express the trigonometrical ratios of the angle $(90° - \theta)$ in terms of those of the angle θ.

Prove the relation $\operatorname{cosec}^6\theta - \cot^6\theta = 1 + 3\operatorname{cosec}^2\theta\cot^2\theta$.

XI B. *December*, 1889.

1. Shew that the angles turned per minute by the hour and minute hand of a watch are respectively 30 minutes and 6 degrees.

Find the angle between the hands of a watch at five minutes past ten o'clock.

2. Find the measures of the three angles of an isosceles triangle in which each of the two equal angles is four times the third angle.

3. Find the values of $\sec 60°$ and $\tan 45°$.
Find values of $\tan\theta$ and $\sin\theta$ which satisfy the equation
$$5(\sec\theta - \tan\theta) = 1.$$

4. Express the trigonometrical ratios of the angle $(90° + \theta)$ in terms of those of the angle θ.
Prove the relation $\sec^6\theta - \tan^6\theta = 1 + 3\sec^2\theta\tan^2\theta$.

XII A. *June*, 1890.

1. Define the cosecant of an angle and prove the formula
$$\operatorname{cosec}^2 A = \cot^2 A + 1.$$
If the slope of a hill be a rise of 1 in 25, find to 3 decimal places, the secant, cosecant and tangent of its inclination to the horizon.

2. Trace the changes in the sign and magnitude of the sine of an angle as the angle increases from 0° to 360°.
Find the angles in the second quadrant whose sines are the same as those of 833° and 1235° respectively.

3. Express the tangent of an angle in terms of the sine.
Prove that $2\sec^2 A = 1 + \sec^4 A - \tan^4 A$.

4. Prove that in any angle ABC
$$AC^2 = AB^2 + BC^2 - 2AB \,.\, BC \,.\, \cos ABC.$$
In a triangle two sides are 10 feet and 17 feet respectively and the angle between these two sides is 30°; find the length of the remaining side.

XII B. *June*, 1890.

1. Define the sine of an angle and prove that
$$\cos^2 A + \sin^2 A = 1.$$
If the slope of a hill be a rise of 1 in 20, find to 3 places of decimals the secant, cosecant and tangent of its inclination to the horizon.

2. Trace the changes in the sign and magnitude of $\cos A°$ as A changes from 0° to 360°.
Find the acute angles whose cosines are the same as 1113° and 762° respectively.

3. Express the cotangent in terms of the cosine.
Prove that $2\operatorname{cosec}^2 A = 1 + \operatorname{cosec}^4 A - \cot^4 A$.

4. Prove that in any triangle the sides are proportional to the sines of the angles opposite them.
In a given triangle one side is 4 feet, the angle opposite to it is 45° and one of the angles adjacent to it is 30°; find the smallest side of the triangle.

ANSWERS TO THE EXAMPLES.

II. Page 5.

1. ·09175 of a right angle = $9^g\ 17'\ 50''$.
2. ·0675 ,, = $6^g\ 75'$.
3. 1·07875 ,, = $107^g\ 87'\ 50''$.
4. ·18042901234567$\dot{9}$ = $18^g\ 4'\ 29''$, etc.
5. 1·467$\dot{7}$,, = $146^g\ 77'\ 77{\cdot}\dot{7}''$.
6. ·5$\dot{4}$,, = $54^g\ 44'\ 44{\cdot}\dot{4}''$.
7. $1°\ 14'\ 15''$.
8. $7°\ 52'\ 30''$.
9. $153°\ 24'\ 29{\cdot}34''$.
10. $21°\ 36'\ 8{\cdot}1''$.
11. $16°\ 12'\ 37{\cdot}26''$.
12. $31°\ 30'$.

III. Page 7.

1. +6.
2. 0.
3. +2.
4. +3.
5. +10.
6. 0.
7. +7.
8. +7.

IV. Pages 15, 16.

1. (i) DA, BD. (ii) DB, AD. (iii) DA, CD. (iv) DC, AD.
2. (i) $\frac{DB}{AB}$. (ii) $\frac{DC}{CA}$. (iii) $\frac{CD}{AD}$. (iv) $\frac{DA}{BA}$. (v) $\frac{DB}{AD}$.
(vi) $\frac{DC}{AC}$. (vii) $\frac{CD}{CA}$. (viii) $\frac{DA}{CD}$. (ix) $\frac{BD}{BA}$. (x) $\frac{DA}{CA}$.
3. (i) $\frac{DB}{CB}$, $\frac{BA}{CA}$. (ii) $\frac{CD}{CB}$, $\frac{CB}{CA}$. (iii) $\frac{DB}{CD}$, $\frac{BA}{CB}$.
(iv) $\frac{DB}{AB}$, $\frac{BC}{AC}$. (v) $\frac{AD}{AB}$, $\frac{AB}{AC}$. (vi) $\frac{DB}{AD}$, $\frac{BC}{AB}$.
4. (i) $\frac{DA}{BA}$. (ii) $\frac{BA}{EA}$ or $\frac{AC}{EC}$. (iii) $\frac{DC}{BC}$. (iv) $\frac{AB}{AE}$.
(v) $\frac{AD}{AB}$ or $\frac{AB}{AC}$. (vi) $\frac{BD}{BC}$. (vii) $\frac{DB}{CD}$, or $\frac{BA}{CB}$, or $\frac{AE}{CA}$.
(viii) $\frac{DA}{BD}$. (ix) $\frac{BA}{EB}$ or $\frac{AC}{EA}$. (x) $\frac{DC}{BD}$. (xi) $\frac{DB}{AB}$ or $\frac{BC}{AC}$.
(xii) $\frac{BE}{AE}$.

IV.

5. $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$; $\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$, $\tan B = \frac{4}{3}$.

7. Of the smaller angle, the sine $= \frac{5}{13}$, cosine $= \frac{12}{13}$, tangent $= \frac{5}{12}$.
Of the larger angle, the sine $= \frac{12}{13}$, cosine $= \frac{5}{13}$, tangent $= \frac{12}{5}$.

8. Of the smaller angle, the sine $= \frac{1}{2}$, cosine $= \frac{\sqrt{3}}{2}$, tangent $= \frac{1}{\sqrt{3}}$.
Of the larger angle, the sine $= \frac{\sqrt{3}}{2}$, cosine $= \frac{1}{2}$, tangent $= \sqrt{3}$.

VI. Page 25.

1. 179 ft. 2. 346 ft. 3. 86·6 ft.
4. 138·5 ft. 5. 63·17 yds. 6. 34·15 ft.

VIII. Page 33.

1. $\sin A = \sqrt{1-\cos^2 A}$, $\tan A = \dfrac{\sqrt{1-\cos^2 A}}{\cos A}$, $\cot A = \dfrac{\cos A}{\sqrt{1-\cos^2 A}}$,
$\sec A = \dfrac{1}{\cos A}$, $\operatorname{cosec} A = \dfrac{1}{\sqrt{1-\cos^2 A}}$.

2. $\sin A = \dfrac{1}{\sqrt{1+\cot^2 A}}$, $\cos A = \dfrac{\cot A}{\sqrt{1+\cot^2 A}}$, $\tan A = \dfrac{1}{\cot A}$,
$\sec A = \dfrac{\sqrt{1+\cot^2 A}}{\cot A}$, $\operatorname{cosec} A = \sqrt{1+\cot^2 A}$.

3. $\sin A = \dfrac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\cos A = \dfrac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A - 1}$,
$\cot A = \dfrac{1}{\sqrt{\sec^2 A - 1}}$, $\operatorname{cosec} A = \dfrac{\sec A}{\sqrt{\sec^2 A - 1}}$.

4. $\sin A = \dfrac{1}{\operatorname{cosec} A}$, $\cos A = \dfrac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec}^2 A}$, $\tan A = \dfrac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$,
$\cot A = \sqrt{\operatorname{cosec}^2 A - 1}$, $\sec A = \dfrac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A - 1}}$.

5. $\cos A = \sqrt{1-\sin^2 A}$, $\tan A = \dfrac{\sin A}{\sqrt{1-\sin^2 A}}$, $\cot A = \dfrac{\sqrt{1-\sin^2 A}}{\sin A}$.
$\sec A = \dfrac{1}{\sqrt{1-\sin^2 A}}$, $\operatorname{cosec} A = \dfrac{1}{\sin A}$.

6. $\sin A = \dfrac{\tan A}{\sqrt{1+\tan^2 A}}$, $\cos A = \dfrac{1}{\sqrt{1+\tan^2 A}}$, $\cot A = \dfrac{1}{\tan A}$,
$\sec A = \sqrt{1+\tan^2 A}$, $\operatorname{cosec} A = \dfrac{\sqrt{1+\tan^2 A}}{\tan A}$.

IX. Page 34.

1. $\frac{3}{4}, \frac{5}{3}$.
2. $\frac{2\sqrt{2}}{3}, \frac{1}{2\sqrt{2}}$.
3. $\frac{4}{5}, \frac{5}{3}$.
4. $\frac{1}{\sqrt{15}}, \frac{\sqrt{15}}{4}$.
5. $\frac{\sqrt{3}}{2}, \frac{1}{2}$.
6. $\frac{\sqrt{5}}{3}, \frac{3}{2}$.
7. $\frac{b}{\sqrt{c^2-b^2}}$.
8. $\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}$.
9. $\frac{\sqrt{a^2-1}}{a}, \frac{1}{\sqrt{a^2-1}}$.
11. $h^2(1+k^2)=1$.

XI. Page 39.

1. The second.
2. The fourth.
3. The second.
4. The third.
5. The fourth.
6. The first.
7. The second.
8. The first.
9. The first.
10. The fourth.
11. The fourth.
12. The first, if n be even, the third, if n be odd.

XII. Page 41.

1. +, +, +.
2. +, −, −.
3. −, −, +.
4. −, +, −.
5. −, +, −.
6. −, −, +.
7. +, −, −.
8. +, +, +.
9. +, +, +.
10. +, +, +.
11. +, −, −.
12. −, +, −.

XIII. Page 43.

1. $+\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$.
2. $+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$.
3. $+\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$.
4. $-\frac{1}{2}, +\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$.
5. $-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, -1$.
6. $+\frac{\sqrt{3}}{2}, +\frac{1}{2}, +\sqrt{3}$.
7. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1$.
8. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1$.
9. $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}$.
10. $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}$.
11. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, +\sqrt{3}$.
12. $-\frac{\sqrt{3}}{2}, +\frac{1}{2}, -\sqrt{3}$.
13. $+\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, +1$.
14. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$.
15. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}$.

XIV. Page 46.

Each of these expressions changes *continuously* as the angle A increases from 0° to 360°, and their values are *repeated* at each complete revolution of the revolving line.

Their values are given below when they are zero and when they cease to increase and begin to decrease, and vice versa. The first table gives also the sign of each between the values.

		0°		90°		180°		270°	
1.	$\cos A$	+1	+	0	−	−1	−	0	+
2.	$\tan A$	0	+	∞	−	0	+	∞	−
3.	$\cot A$	∞	+	0	−	∞	+	0	−
4.	$\sec A$	+1	+	∞	−	−1	−	∞	+
5.	$\operatorname{cosec} A$	∞	+	+1	+	∞	−	−1	−
6.	$1-\sin A$	+1	+	0	+	+1	+	0	+
7.	$\sin^2 A$	0	+	+1	+	0	+	+1	+

		0°	45°	90°	135°	180°	225°	270°	315°
8.	$\sin A \cdot \cos A$	0	$+\frac{1}{2}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$-\frac{1}{2}$
9.	$\sin A+\cos A$	+1	$+\sqrt{2}$	+1	0	−1	$-\sqrt{2}$	−1	0
10.	$\tan A+\cot A$	∞	+2	∞	−2	∞	+2	∞	−2
11.	$\sin A-\cos A$	−1	0	+1	$+\sqrt{2}$	+1	0	−1	$-\sqrt{2}$

The following figures exhibit the changes in the sign and magnitude of sin θ (fig. i.), cos θ (fig. ii.), and tan θ (fig. iii.). The measure of the distance from 0 along the line 0X = the circular measure of the angle; the vertical distance from 0X measures the Ratio.

Fig. i. The Curve of the Sine.

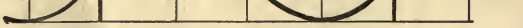


Fig. ii. The Curve of the Cosine.

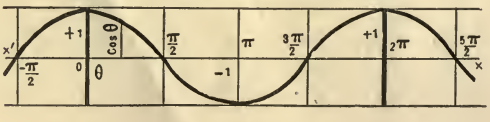
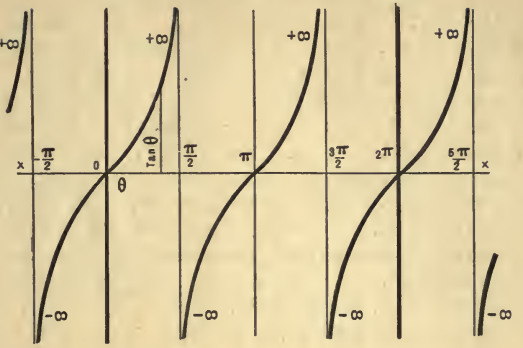


Fig. iii. The Curve of the Tangent.



XV. Page 48.

1. (i) 30°, 150°, −210°, −330°. (ii) 45°, 135°, −225°, −315°.
(iii) 60°, 120°, −240°, −300°. (iv) −30°, −150°, 210°, 330°.

2. (i) 20°, 160°, 380°, 520°. (ii) $\frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}$.

(iii) $\frac{8\pi}{7}, \frac{13\pi}{7}, \frac{22\pi}{7}, \frac{27\pi}{7}$.

3. (i) $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$. (ii) $\theta = n\pi + (-1)^n\frac{\pi}{6}$.

(iii) $n\pi \pm \frac{\pi}{4}$.

XVI. Pages 51, 52.

1. (i) $\theta = 2n\pi \pm \frac{1}{3}\pi$. (ii) $\theta = n\pi + \frac{1}{4}\pi$. (iii) $\theta = n\pi - \frac{1}{4}\pi$.
(iv) $\theta = n\pi - \frac{1}{3}\pi$. (v) $\theta = 2n\pi \pm \frac{4}{5}\pi$. (vi) $\theta = n\pi + \frac{3}{4}\pi$.

3. The tangent. 4. No.

5. (i) 60°, −60°, 300°, −300°. (ii) 45°, 225°, 405°, −135°.
(iii) −45°, −225°, 135°, 315°. (iv) −60°, −240°, 120°, 300°.
(v) 144°, −144°, 216°, −216°. (vi) 135°, −45°, −225°, 315°.

XX. Page 64.

In the following answers n stands for '*any whole number.*'

1. (*a*) 45°, (*b*) $n \times 180° + (-1)^n 45°$. 2. (*a*) $\pm 30°$,
(*b*) $n \times 180° \pm 30°$. 3. (*a*) 45°; or, 135°, (*b*) $n \times 180° \pm 45°$.

XX.

4. (*a*) $\pm 60°$, (*b*) $n \times 180° \pm 60°$. **5.** (*a*) $\pm 30°$, (*b*) $n \times 180° \pm 30°$. **6.** (*a*) $\pm 30°$, (*b*) $n \times 180° \pm 30°$. **7.** (*a*) 30°, (*b*) $n \times 180° + (-1)^n 30°$. **8.** (*a*) 0; or, 45°, (*b*) $n \times 180°$; or, $n \times 360° \pm 45°$. **9.** (*a*) 90°; or, 60°, (*b*) $(2n+1) \times 90°$; or, $n \times 180° + (-1)^n 60°$. **10.** (*a*) $\pm 60°$, (*b*) $n \times 180° \pm 60°$. **11.** (*a*) 45°, (*b*) $n \times 180° + 45°$; or $n \times 180°$ + the angle whose tangent is 3. **12.** (*a*) 45°, (*b*) $n \times 180° + 45°$. **13.** (*a*) 90°; or, 45°, (*b*) $(2n+1) \times 90°$; or, $2n \times 180° \pm 45°$. **14.** (*a*) $-90°$, (*b*) $2n \times 180° - 90°$; or, $n \times 180° + (-1)^n \times$ an angle whose sine is $\frac{1}{3}$. **15.** (*a*) $\pm 45°$, (*b*) $n \times 180° \pm 45°$. **16.** (*a*) 45°, (*b*) $n \times 180° + (-1)^n 45°$.
17. (*a*) 30°, (*b*) $2n \times 180° \pm 30°$.
18. (*a*) 30°, (*b*) $n \times 180° + (-1)^n 30°$.

XXVII. Pages 78, 79.

1. $\cos A = \frac{1}{2}$, $\cos \frac{1}{2}A = \frac{1}{2}\sqrt{3}$. **2.** 45°, 60°, 75°.
3. 135°, 30°, 15°. **4.** 3. **5.** 14. **6.** $1+\sqrt{3}$. **7.** 120°.
8. 120°. **9.** 120°. **10.** 90°, 36° 52′. **11.** 130° 27′. **12.** 125° 6′.
13. 120°. **14.** $A = 54°$ or 126°, $B = 108°$ or 36°. **15.** $a = 1$.
16. $C = 30°$, $a = \sqrt{3}+1$, $b = 2$. **17.** $A = 75°$, $a = b = \sqrt{3}+1$.
18. $C = 60°$ or 120°. **19.** $100\sqrt{3}$. **20.** No.
22. $A = 105°$, $C = 60°$, $B = 15°$. **23.** $\frac{1}{2}\sqrt{3}(\sqrt{5}+1)$.
24. $A = 90°$ or 60°, $C = 75°$ or 105°, $a = 2\sqrt{2}$ or $\sqrt{6}$. **25.** 30° or 150°. **26.** $A = 45°$ or 135°, $B = 30°$ or 120°, $b = \sqrt{2}(1+\sqrt{3})$ or $\sqrt{6}(1+\sqrt{3})$. **27.** 60°, 75°, 6 yds. **28.** It is impossible.
30. $15 : 8\sqrt{3} : 4\sqrt{5}+6$.

HINTS AND ANSWERS TO EXAMINATION PAPERS.

I A. Oct. 86. **1.** $\frac{1}{60}$ of $\frac{1}{12}$ of $360 \times 60' = 30'$.

2. (i) $\frac{MP^2}{OP^2} + \frac{OM^2}{MP^2} + \frac{OM^2}{OP^2} = 1 + \frac{OM^2}{MP^2} = \frac{MP^2 + OM^2}{MP^2} = \frac{OP^2}{MP^2}$. (ii) Multiply each fraction in the second part by $\cos A$ and we have to prove that $\frac{\sin A + 1 + \cos A}{\sin A + 1 - \cos A} = \frac{1 + \cos A}{\sin A}$, which is true if, $\sin^2 A + \sin A(1 + \cos A) = \sin A(1 + \cos A) + 1 - \cos^2 A$. **3.** In fig. on p. 19 draw MD perpen-

dicular to OP then $MD=20$ ft. $\therefore$ $PM=40$ ft. and $OP=40$ ft. $\times\frac{2}{\sqrt{3}}$ $=\frac{80}{3}\times\sqrt{3}=46{\cdot}18\ldots$ ft. and $OP=AB$ in question. **4.** When $\sin\theta=\frac{1}{2}\sqrt{3}$ then $\theta=n\times180°+(-1)^n 60°$ $\therefore$ $\theta=60°$; or, $120°$; or, $360°+60°$ or $540°-60°$ $\therefore$ $\frac{1}{2}A=30°$; or, $60°$; or, $210°$; or, $240°$ etc. $\therefore$ $\sin\frac{1}{2}\theta=\frac{1}{2}$; or, $\frac{1}{2}\sqrt{3}$; or, $-\frac{1}{2}$; or, $-\frac{1}{2}\sqrt{3}$.

I B. Oct. 86. 1. $30''$. **3.** $\frac{1}{3}$ of $24\times\sqrt{3}=13{\cdot}856$. **4.** $\pm\frac{1}{2}\sqrt{3}$, $\pm\frac{1}{2}$.

II A. Dec. 86. 1. (i) $\frac{OP}{MP}\times\frac{OP}{OM}=\frac{MP^2+OM^2}{MP\,.\,OM}=\frac{MP}{OM}+\frac{OM}{MP}$. (ii) the exp. multiplied by $\cos^2\theta\,(1+\sin\theta)\,(1+\sec\theta)$ is $\sin^2\theta\,(\sec^2\theta-1)$ $+(\sin^2\theta-1)=1-\sin^2\theta+\sin^2\theta-1=0$. **2.** $\frac{MP}{OP}\cdot\frac{MP}{OM}=\frac{OP^2-OM^2}{OP\,.\,OM}$ $=\frac{OP}{OM}-\frac{OM}{OP}=\sec A-\cos A$. (ii) The left hand exp. is $\frac{1}{8}(\sin^2A+\cos^2A)$ $(\cos^4A-\cos^2A\,\sin^2A+\sin^4A)-\frac{1}{4}\{(\cos^4A+\sin^4A)-2\sin^2A\cos^2A\}$ $=(\frac{1}{8}-\frac{1}{4})(\cos^4A+\sin^4A)+(\frac{1}{2}-\frac{1}{8})\sin^2A\cos^2A=\frac{1}{16}(\cos^4A+\sin^4A$ $+2\sin^2A\cos^2A)=\frac{1}{16}(\sin^2A+\cos^2A)^2$. **3.** (One diagonal)$^2=15^2+24^2$ $-2\times15\times24\cos60°$ [Art. 91] $=225+596-360=441=(21)^2$ the (other diagonal)$^2=15^2+24^2-2\times15\times24\cos120°=1161=(33{\cdot}9\ldots)^2$.

II B. Dec. 86. 4. $\sqrt{499}$, $\sqrt{1399}$.

III A. June 87. 1. $\frac{1}{60}\times\frac{1}{60}\times\frac{1}{24}\times360\times60\times60''=15''$. **2.** $\sin A=\sqrt{(1-\cos^2A)}=\sqrt{(1-m^2)}$; $\tan A=\frac{\sqrt{(1-m^2)}}{m}$. We have to prove that $(1-\tan^2B)\cos^2B=\cos^2B-\sin^2B$. **3.** $2=2\sin^2A+\cos A$; or, $\cos A\,(1-2\cos A)=0$ $\therefore$ $\cos A=0$; or, $\cos A=\frac{1}{2}$ $\therefore$ $A=$ any odd multiple of $90°$; or $2n\times180°\pm60°$. **4.** See fig. Art. 67. The tangents of the angles of the figure are equal in magnitude but of opposite sign.

III B. June 87. 1. $15'$. **2.** $\cos A=\sqrt{(1-m^2)}$; $\tan A=m\,(\sqrt{1-m^2})$. **3.** $A=n\times180°$; or, $n\times180°+(-1)^n 30°$.

IV A. Oct. 87. 1. Let the second and third angles be x and $2x$ respectively, then $x+2x+80°\,12'\,45''=180°$, whence $x=33°\,.\,15'\,.\,45''$. **2.** $\sin\theta=\sqrt{(1-\cos^2\theta)}=\sqrt{\{1-1/\sec^2\theta\}}$; $\tan\theta=\sin\theta/\cos\theta$ $=\sqrt{\{1-1/\sec^2\theta\}}\sec\theta$. Multiplying both sides of proposed identity by $\sin^4\theta\cos^4\theta$, we have to prove that $(1-\sin^4\theta)\,(1-\cos^4\theta)=\sin^4\theta\cos^4\theta$ $+2\sin^2\theta\cos^2\theta$; or that $\cos^2\theta\,(1+\sin^2\theta)\sin^2\theta\,(1+\cos^2\theta)=\sin^2\theta\cos^2\theta$ $(2+\sin^2\theta\cos^2\theta)$ which is true. **3.** $\sin^2x°+1=\frac{5}{2}\sin x°$; or

$(\sin x° - \frac{5}{4})^2 - \frac{9}{16} = 0$; or $(\sin x° - \frac{1}{2})(\sin x° - 2) = 0$; $\therefore \sin x° = \frac{1}{2}$; $x° = 30°$. **4.** The angle whose sine is $> \frac{1}{2}$ must lie between $n \times 360° + 30°$ and $n \times 360° + 120°$. The angle whose cosine is $> \frac{1}{2}\sqrt{2}$ must lie between $n \times 360° - 45°$ and $n \times 360° + 45°$ $\therefore$ the required angle lies between $n \times 360° + 45°$ and $n \times 360° + 60°$.

IV B. Oct. 87. **1.** 33°. 15′. 12″; 99°. 45′. 36″. **3.** 60°. **4.** between $n \times 360° + 45°$ and $n \times 360° + 60°$.

V A. Dec. 87. **1.** The cosine lies between $+1$ and -1; the cosecant between $-\infty$ and -1 and between $+1$ and $+\infty$; $\cos\theta = \sqrt{(1 - 1/\operatorname{cosec}^2\theta)}$; $\operatorname{cosec}\theta = 1/\sqrt{(1 - \cos^2\theta)}$. **2.** $\sin(4A + 3B) = 1$ and $\sin(3A + B) = \frac{1}{2}$ $\therefore 4A + 3B = 90°$ and $3A \times B = 30°$ $\therefore A = 0°$ and $B = 30°$; [or, generally, $4A + 3B = (2n + 1)\,90°$; $3A + B = n \times 180° + (-1)^n\,30°$]. **3.** See Art. 77. **4.** $3\tan A + 2 = 3\sec A$ $\therefore 9\tan^2 A + 12\tan A + 4 = 9\sec^2 A = 9 + 9\tan^2 A$ $\therefore$ [$\tan A = \infty$; or,] $\tan A = \frac{5}{12}$.

V B. Dec. 87. **2.** $A = 36°$; $B = 48°$. **4.** $\cot A = \infty$; or, $\cot A = \frac{5}{12}$.

VI A. June 88. **1.** Multiply both sides by $\sin^2\theta \cos^2\theta$. **2.** $\cos 17° > \sin 17°$ for $\sin 45° = \cos 45°$ and as the angle diminishes the sine diminishes while the cosine increases; hence also $\sec 17° < \operatorname{cosec} 17°$ and all secants and all cosecants are greater than 1, while all sines and all cosines are less than 1. Hence the order is $\operatorname{cosec} 17° > \sec 17° > \cos 17° > \sin 17°$. As an angle θ increases from 0 to 90°, $\cos\theta$ diminishes *continuously* from 1 to 0 and $\tan\theta$ increases *continuously* from 0 to ∞; $\tan\theta = \cos\theta$; $\therefore \sin\theta = 1 - \sin^2\theta$ $\therefore \sin\theta = \frac{1}{2}\sqrt{5} - \frac{1}{2} = \frac{1}{2}(2{\cdot}236\ldots) - {\cdot}5 = {\cdot}618\ldots$. **3.** $\sin(180° - A) = \sin(90° + 90° - A) = \cos(90° - A) = \sin A$. **4.** $\theta = n \times 180° + 30°$ $\therefore \theta = 30°$, 210°, 390°, 570°, 750°, 930°. **5.** $\sin\alpha = \sqrt{2}\sin\beta$; $\sqrt{2}\cos\beta = \sqrt{3}\cos\alpha$; $\therefore \sin^2\alpha = 2(1 - \frac{3}{2}\cos^2\alpha)$ $\therefore \sin^2\alpha = \frac{1}{2}$; $\therefore \sin\alpha = \frac{1}{2}\sqrt{2}$; $\sin\beta = \frac{1}{2}$; $\therefore \alpha = 45°$; $\beta = 30°$.

VI B. June 88. **2.** $\sec 57° > \operatorname{cosec} 57° > \sin 57° > \cos 57°$; ·618… **4.** 30°, 210°, 390°, 570°, 750°, 930°. **5.** $\alpha = 45°$; $\beta = 30°$.

VII A. Oct. 88. **1.** $\frac{OP}{OM} \times \frac{OP}{MP} = \frac{OM^2 + MP^2}{OM \times MP} = \frac{OM}{MP} + \frac{MP}{OM}$. $\operatorname{Sec} A = \operatorname{cosec} A/\sqrt{(\operatorname{cosec}^2 A - 1)}$. **2.** $A° = n \times 180° + (-1)^n(-60°)$; $A = 600°$, 660°, 960°. **3.** $\alpha = 90° - (180° - A) = A - 90°$, $\beta = 180° - (90° - A) = 90° + A$ $\therefore \alpha + 180° = \beta$; $\therefore \sin\beta = \sin(180° + \alpha) = \cos(90° + \alpha) = -\sin\alpha$ $\tan\beta = \tan(180° + \alpha) = \tan\alpha$; by fig. on page 50.

VII B. Oct. 88. **2.** 1020°, 1260°, 1320°.

VIII A. Dec. 88. **1.** Let the number of sides be $2n$ and n; the angles are $(2n \times 180° - 360°)/2n$ and $(n \times 180° - 360°)/n$. $\therefore 2n \times 180° - 360° = 3(n \times 180° - 360°)$ $\therefore n = 4$ $\therefore$ the angles are 90° and 135°. **2.** The left hand exp. $= (1 - \sin^4 A - \cos^4 A)/\sin^2 A \cos^2 A = \{(\sin^2 A + \cos^2 A)^2 - \sin^4 A - \cos^4 A\}/\sin^2 A \cos^2 A = 2 \sin^2 A \cdot \cos^2 A/\sin^2 A \cos^2 A = 2$.
4. $-\frac{1}{2}$, $-\frac{1}{3}\sqrt{3}$. Here, $\sin\theta = \pm\frac{1}{2}\sqrt{3}$ $\therefore \theta = n \times 180° \pm 60°$; 60°, 120°, 240°, 300°, 420°, 480°, 600°, 660°, 780°, 840°, 960°.
5. (i) $5\tan^2 A° + 6 = 11\tan A°$; or $(5\tan A° - 6)(\tan A° - 1) = 0$ $\therefore A = 45°$. (ii) $2\cos^2 A° + 3\sin^2 A° = \frac{11}{4}$ $\therefore \sin^2 A° = \frac{3}{4}$ $\therefore A = 60$.

VIII B. Dec. 88. **1.** 108°, 144°. **4.** $+\frac{1}{2}$, $-1\frac{1}{3}\sqrt{3}$. 30°, 150°, 210°, 330°, 390°, 510°, 570°, 690°, 750°, 870°, 930°. **5.** (i) 45. (ii) 60.

IX A. June 89. **1.** $(28°.57'.54'')/(1°.4'.22'') = (14°.28'.57'')/(32'.11'') = 52137''/1931'' = 27$. **2.** The left hand exp. $= (\tan A + \cot A)^2 - \sec^2 A - 1 = \tan^2 A + 2 + \cot^2 A - \sec^2 A - 1 = (\tan^2 A + 1) - \sec^2 A + \cot^2 A = \cot^2 A$. **3.** At 2 o'clock the hour hand is 60° past 12; at a quarter past two it is $60° + \frac{1}{4}$ of $\frac{1}{12}$ of $360° = 67\frac{1}{2}°$. At a quarter past two the minute hand is 90° past 12 $\therefore$ the angle between them is $22\frac{1}{2}°$. **4.** $\frac{7}{8} = \cdot875$; $\cos 30° = \frac{1}{2}\sqrt{3} = \cdot866\ldots$; the cosine diminishes as the angle increases, $\therefore$ this angle is $< 30°$.

IX B. June 89. **1.** 21. **3.** $22\frac{1}{2}°$. **4.** less.

X A. Oct. 89.
1. $4 - 4\sin^2\theta - 8\sin\theta + 1 = 0$ or $(2\sin\theta - 1)(2\sin\theta + 5) = 0$ $\therefore \sin\theta = \frac{1}{2}$; $\therefore \theta = n \times 180° + (-1)^n 30°$. **2.** $\frac{1}{3}\sqrt{3} = \cdot577\ldots$; $\cos 45° = \frac{1}{2}\sqrt{2} = \cdot707\ldots$; $\cos 60° = \cdot5$ and $\cos\theta$ diminishes as θ increases. **3.** $\tan\theta = \sqrt{(1 - \cos^2\theta)}/\cos\theta = 1/\sqrt{(\operatorname{cosec}^2\theta - 1)}$ $\cos\theta = 1/\sqrt{(1 + \tan^2\theta)} = 1 \div \sqrt{5}$; $\operatorname{cosec}\theta = \sqrt{(1 + \tan^2\theta)}/\tan\theta = \frac{1}{2}\sqrt{5}$. **4.** $\sin(180° + A) = -\sin A$; $\cos(180° + A) = -\cos A$, $\tan(180° + A) = \tan A$; when $\sin A° = -\frac{1}{2}\sqrt{3}$ $A = n \times 180 + (-1)^n(-60)$. Hence $A = 600$, $\cos 600° = -\frac{1}{2}$; 660, $\cos 660° = \frac{1}{2}$; 960, $\cos 960° = -\frac{1}{2}$.

X B. Oct. 89. **1.** $n \times 360° \pm 30°$. **3.** $\tan\theta = \sin\theta/(1 - \sin^2\theta) = \sqrt{(\sec^2\theta - 1)}$. **4.** 510, $\sin 510° = \frac{1}{2}$; 570, $\sin 570° = -\frac{1}{2}$; 870, $\sin 870° = \frac{1}{2}$; 930, $\sin 930° = -\frac{1}{2}$.

XI A. Dec. 89. **1.** (i) The times are $\frac{1}{360}$ of $60 \times 60''$; or $10''$ and $\frac{1}{360}$ of $12 \times 60'$; or $2'$. (ii) The minute hand is $\frac{4}{60}$ of 360° to 12 = 24° to 12; the hour hand is $(\frac{1}{60}\frac{8}{0}$ of $360° - \frac{1}{12}$ of $\frac{4}{60}$ of $360°)$ past 12 or 58° past 12. $\therefore$ the angle is $(24° + 58°)$. **2.** Let the equal angles each contain $x°$ then $x° + x° + 4x° = 180°$ $\therefore x = 30$; the angles are 30°, 30° and 120°.

3. $9\sec^2\theta = 4 + 12\tan\theta + 9\tan^2\theta$; or, $\tan\theta = \frac{5}{12}$, $\sec\theta = \frac{13}{12}$.
4. $\sin(90° - \theta) = \cos\theta$; $\cos(90° - \theta) = \sin\theta$, $\tan(90° - \theta) = \cot\theta$. $\operatorname{cosec}^6\theta - \cot^6\theta = (\operatorname{cosec}^2\theta - \cot^2\theta)(\cot^4\theta + \operatorname{cosec}^2\theta\cot^2\theta + \cot^4\theta)$ $= (\operatorname{cosec}^2\theta - \cot^2\theta)^2 + 3\operatorname{cosec}^2\theta\cot^2\theta = 1 + 3\operatorname{cosec}^2\theta\cot^2\theta$, because $\operatorname{cosec}^2\theta - \cot^2\theta = 1$.

XI B. Dec. 89. 1. $90° - 2\frac{1}{2}°$. **2.** 80°, 80°, 20°. **3.** $\tan\theta = \frac{12}{5}$; $\sec\theta = \frac{13}{5}$. **4.** $\sin(90° + \theta) = \cos\theta$; $\cos(90° + \theta) = -\sin\theta$; $\tan(90° + \theta) = -\cot\theta$.

XII A. June 90. 1. When the hypotenuse is 25 and the perpendicular 1, the base is $\sqrt{(25^2 - 1)} = \sqrt{624} = 24{\cdot}98\ldots$ Hence the secant is $\frac{25}{24{\cdot}98} = 1{\cdot}0008$. The cosecant is $\frac{1}{25} = {\cdot}04$. The tangent is $\frac{1}{24{\cdot}68}$ $= {\cdot}040\ldots$. **2.** $833° = 2 \times 360° + 113°$. The required angle is 113°; $1235° = 3 \times 360° + 227° = 3 \times 360° + 180° + 47°$ and $\therefore \sin 1235°$ $= \sin(180° + 47°) = -\sin(180° - 47°)$. **3.** $\sec^4 A - \tan^4 A + 1 = \sec^4 A$ $- (\sec^2 A - 1)^2 + 1 = 2\sec^2 A$. **4.** $= \sqrt{(17^2 + 10^2 - 2 \times 10 \times 17 \times \cos 30°)}$ $= \sqrt{(389 - 170 \times \sqrt{3})} = \sqrt{(389 - 170 \times 1{\cdot}7320\ldots)} = \sqrt{94{\cdot}56} = 9{\cdot}7\ldots$.

XII B. June 90. 1. $\sec = 1{\cdot}0015\ldots$ $\operatorname{cosec} = {\cdot}05$, $\tan = {\cdot}050\ldots$
2. $\cos 15° = -\cos 1113° \cos 60° = \cos 762°$. **4.** $2\sqrt{2}$ feet.

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